

HOMOLOGICAL INVARIANTS AND QUASI-ISOMETRY

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ABSTRACT. Building upon work of Y. Shalom we give a homological-algebra flavored definition of an induction map in group homology associated to a topological coupling. As an application we obtain that the cohomological dimension cd_R over a commutative ring R satisfies the inequality $\mathrm{cd}_R(\Lambda) \leq \mathrm{cd}_R(\Gamma)$ if Λ embeds uniformly into Γ and $\mathrm{cd}_R(\Lambda) < \infty$ holds. Another consequence of our results is that the Hirsch ranks of quasi-isometric solvable groups coincide. Further, it is shown that the real cohomology rings of quasi-isometric nilpotent groups are isomorphic as graded rings. On the analytic side, we apply the induction technique to Novikov-Shubin invariants of amenable groups, which can be seen as homological invariants, and show their invariance under quasi-isometry.

1. INTRODUCTION AND STATEMENT OF RESULTS

What is the relation between the (co)homology groups of two given quasi-isometric groups? This question is as natural as challenging since a quasi-isometry, being a purely geometric notion, provides no obvious way to produce a reasonable map in group cohomology. Recently, Yehuda Shalom [Sha] introduced a whole new circle of ideas and techniques into the study of the geometry of amenable groups. These techniques involve representation theory and cohomology. The starting point is an induction map in group cohomology (for both ordinary and reduced) associated to a topological coupling of quasi-isometric groups (defined in 2.2). The existence of such a topological coupling is equivalent to being quasi-isometric due to Gromov's dynamic characterization of quasi-isometry [Gro93, 0.2.C₂']. Restricting to ordinary cohomology, Shalom's induction map is available for the more general situation of uniform embeddings. It is defined explicitly in terms of the standard homogeneous resolution.

We provide a different and more abstract approach to the induction which works for ordinary cohomology *and* homology, is natural in the coefficients and compatible with cup- and cap-products. In doing so we build heavily upon the ideas of Shalom. Let us discuss now the main results of the paper. We first recall the notion of uniform embedding, a notion encompassing subgroup inclusions and quasi-isometric embeddings.

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Definition 1.1. Let Λ, Γ be discrete countable groups.

- (i) A map $\phi : \Lambda \rightarrow \Gamma$ is called a **uniform embedding** if for every sequence of pairs $(\alpha_i, \beta_i) \in \Lambda \times \Lambda$ one has:

$$\alpha_i^{-1}\beta_i \rightarrow \infty \text{ in } \Lambda \iff \phi(\alpha_i)^{-1}\phi(\beta_i) \rightarrow \infty \text{ in } \Gamma.$$

Here $\rightarrow \infty$ means eventually leaving every finite subset.

- (ii) The groups Λ, Γ are called **quasi-isometric** if there exists a uniform embedding $\phi : \Lambda \rightarrow \Gamma$ and finite subset $C \subset \Gamma$ such that $\phi(\Lambda) \cdot C = \Gamma$.

That the latter definition of quasi-isometry is equivalent to the usual one (see definition 2.1) is part of theorem 2.2. Now suppose that Λ uniformly embeds into Γ , and let R be a commutative ring. What one finally gets (theorem 3.2), taking a long route via Gromov's dynamic criterion 2.2, is the following: There is a compact topological space Y with a continuous Λ -action, a functor $\bar{I} : \{R\Lambda\text{-modules}\} \rightarrow \{R\Gamma\text{-modules}\}$ and a homomorphism, called the **induction**, in cohomology

$$(1.1) \quad I^n : H^n(\Lambda, M) \longrightarrow H^n(\Lambda, \mathcal{F}(Y; R) \otimes_R M) \xrightarrow{\cong} H^n(\Gamma, \bar{I}(M))$$

for every $R\Lambda$ -module M . Here $\mathcal{F}(Y; R)$ is the ring of functions $Y \rightarrow R$ with the property that the preimage of any $r \in R$ is open and closed; it carries a natural Λ -action. We consider $\mathcal{F}(Y; R) \otimes_R M$ as an $R\Lambda$ -module by the diagonal Λ -action. The first map in (1.1) is induced by the inclusion $M \hookrightarrow \mathcal{F}(Y; R) \otimes_R M$, $m \mapsto \text{id}_Y \otimes m$. The second map in (1.1) is always an isomorphism. If we can prove that the first map and hence I^n are injective under certain assumptions, then we get the estimate $\text{cd}_R(\Lambda) \leq \text{cd}_R(\Gamma)$ for the cohomological dimensions over R . So we do not need to care so much about the actual definitions of $\bar{I}(M)$ and the second map in (1.1). The theorem we obtain by analyzing the first map is the following (shown in section 4).

Theorem 1.2. *Let R be a commutative ring, and suppose Λ embeds uniformly into Γ where Λ and Γ are discrete, countable groups. Then the following two statements hold.*

- (i) *If $\text{cd}_R(\Lambda)$ is finite, then we have $\text{cd}_R(\Lambda) \leq \text{cd}_R(\Gamma)$.*
- (ii) *If Λ is amenable and $\mathbb{Q} \subset R$, then we have $\text{cd}_R(\Lambda) \leq \text{cd}_R(\Gamma)$.*

Furthermore, (i), (ii) hold true if cd_R is replaced by the homological dimension hd_R .

Here statement (ii) for the cohomological dimension is already proved in [Sha, theorem 1.5] and was conjectured for the homological dimension in [Sha, section 6.4]. An important point is that we can deal with non-amenable groups by imposing a finiteness condition. The theorem above also generalizes a result of Gersten [Ger93].

By a result of Stammbach [Sta70], the rational homological dimension of a solvable group equals its Hirsch number. Hence we obtain the following corollary which was known before only under additional finiteness conditions on the groups (see [BG96], [Sha]).

Corollary 1.3. *Let Γ be a solvable group, and let Λ be a solvable group quasi-isometric to Γ . Then the Hirsch ranks of Γ and Λ coincide.*

Recall that the Hirsch rank $h(\Gamma)$ of a solvable group Γ is defined as the sum

$$h(\Gamma) = \sum_{i \geq 0} \dim_{\mathbb{Q}}(\Gamma^i / \Gamma^{i+1} \otimes_{\mathbb{Z}} \mathbb{Q}),$$

where $\Gamma^{(i)}$ is the i -th term in the derived series of Γ . An interesting application of 1.2 (i) was pointed out to me by Shalom. Let Γ, Λ two non-uniform arithmetic lattices in the same connected semisimple Lie group G with finite center. Conjecturally [Sha, section 6.1], Λ uniformly embeds into Γ if and only if $\text{Q-rank } \Lambda \geq \text{Q-rank } \Gamma$, with equality if and only if Λ, Γ are commensurable. We remark that if Γ is uniform, i.e. $\text{Q-rank } \Gamma = 0$, then every discrete subgroup of G embeds uniformly into Γ . Then 1.2 (i) together with a theorem of Borel and Serre [BS73], which expresses $\text{cd}_{\mathbb{Q}}(\Gamma)$ of a lattice Γ as the difference of the dimension of the associated symmetric space with the Q-rank of Γ , imply the "only if"-statement:

Corollary 1.4. *Let Γ, Λ be arithmetic lattices in the same connected semisimple Lie group with finite center. If Λ uniformly embeds into Γ then $\text{Q-rank } \Lambda \geq \text{Q-rank } \Gamma$ holds.*

We remark that irreducible lattices in semisimple Lie groups of \mathbb{R} -rank ≥ 2 are arithmetic by Margulis' arithmeticity theorem.

An interesting generalization of Shalom's theorem saying that the Betti numbers of quasi-isometric nilpotent groups coincide [Sha, theorem 1.2] is proved in section 5.2:

Theorem 1.5. *If Γ and Λ are quasi-isometric nilpotent groups, then the real cohomology rings $H^*(\Gamma, \mathbb{R})$ and $H^*(\Lambda, \mathbb{R})$ are isomorphic as graded rings.*

By a theorem of Malcev any finitely generated torsion-free nilpotent group Γ is discretely and cocompactly embedded in a unique simply connected nilpotent Lie group G , the so-called (*real*) *Malcev completion* of Γ . Thus Γ has an associated real Lie algebra \mathfrak{g} . By [Nom54, theorem 1] the cohomology algebras of \mathfrak{g} and Γ are isomorphic:

$$H^*(\Gamma, \mathbb{R}) \cong H^*(\mathfrak{g}, \mathbb{R}).$$

It is a long standing question whether the real Malcev completions of quasi-isometric nilpotent groups are isomorphic. Note that a positive answer would imply the preceding theorem. We remark that the *graded* Lie algebra associated to \mathfrak{g} is a quasi-isometry invariant of Γ by a deep theorem of Pierre Pansu [Pan89].

On the analytic side, we apply our methods to **Novikov-Shubin invariants** of amenable groups, i.e. of the classifying spaces of these groups. The i -th Novikov-Shubin invariant, defined by Novikov and Shubin in the 80's [NS86b], [NS86a], can be seen as a kind of secondary information associated to the i -th L^2 -Betti number. In the Riemannian setting, the i -th Novikov-Shubin invariant $\alpha_i(\tilde{M})$ of the universal covering \tilde{M} of a compact Riemannian manifold M quantifies the speed of the convergence of the limit

$$b_i^{(2)}(\tilde{M}) = \lim_{t \rightarrow \infty} \underbrace{\int_{\mathcal{F}} \text{tr}_{\mathbb{R}} \left(e^{-t\Delta_i}(x, x) \right) d\text{vol}_x}_{=\theta_i(t)}.$$

Here \mathcal{F} is a fundamental domain for the $\pi_1(M)$ -action on \tilde{M} , Δ_i is the Laplacian on the i -forms on \tilde{M} , and $b_i^{(2)}(\tilde{M})$ denotes the i -th L^2 -Betti number of \tilde{M} . For instance, if the integral $\theta_i(t)$ has an asymptotic behavior like $t^{-p} + b_i^{(2)}(\tilde{M})$ for $t \rightarrow \infty$, then $\alpha_i(\tilde{M})$ would be equal to p . As for $b_i^{(2)}(X)$, there is a notion of the Novikov-Shubin invariant $\alpha_i(X)$ of a finite type (i.e. finitely many Γ -cells in each dimension) free Γ -CW complex X , which coincides with the heat kernel definition in the case of universal coverings of compact Riemannian manifolds. For more information see [Lüc02, chapter 2]. The Novikov-Shubin invariants $\alpha_i(\Gamma)$, $i \geq 1$, of a group Γ are defined as the Novikov-Shubin invariants of the classifying space $E\Gamma$ provided $E\Gamma$ admits a model of finite type. Their relation to the geometry of groups is already indicated by the value of $\alpha_1(\Gamma)$ for finitely generated Γ (see the computation in [LRS99, proposition 3.2], based on results of Varopoulos).

$$\alpha_1(\Gamma) = \begin{cases} \infty^+ & \text{if } \Gamma \text{ is finite or non-amenable,} \\ n & \text{if } \Gamma \text{ has polynomial growth of degree } n, \\ \infty & \text{otherwise.} \end{cases}$$

We briefly dwell on the definition of $\alpha_i(\Gamma)$ we will actually work with. This definition of $\alpha_i(\Gamma)$, developed in [LRS99], interprets $\alpha_i(\Gamma)$ as an *invariant of the group homology* of Γ with coefficients in the group von Neumann algebra $\mathcal{N}(\Gamma)$, and is available for any group, not only for those with a finite type classifying space. The point of view is similar as in the algebraic definition of L^2 -Betti numbers by Wolfgang Lück [Lüc98a], [Lüc98b], and so is the motivation. Here is a typical situation: A group Γ to which the original definition of $\alpha_i(\Gamma)$ applies could have a normal subgroup Λ to which it does not apply, but for which we know the value or an estimate of $\alpha_i(\Lambda)$ (extended definition). Then this information tells us something about $\alpha_i(\Gamma)$ by using the Hochschild-Serre spectral sequence (cf. [LRS99]).

However, as opposed to L^2 -Betti numbers, this extension to all groups does not preserve all the properties of α_i one wants to have; the maximal subclass \mathcal{CM} of amenable groups for which we get a reasonable notion of α_i is described in definition 6.7. The class \mathcal{CM} contains inter alia all amenable groups of type FP_∞ over \mathbb{C} , hence including the class of amenable groups for which α_i was originally defined, and is closed under quasi-isometry.

Theorem 1.6. *Let $\Gamma \in \mathcal{CM}$. If Λ is quasi-isometric to Γ , then $\alpha_i(\Gamma) = \alpha_i(\Lambda)$ for $i \geq 1$.*

According to [Gro93, 8.A6] it is not unreasonable to expect that the quasi-isometry invariance of α_i holds true for, at least, all the groups to which the classical definition of Novikov-Shubin invariants applies.

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2. QUASI-ISOMETRY AND TOPOLOGICAL COUPLINGS

This technical section lays the basis for the whole paper. As in Shalom's work on the geometry of amenable groups our starting point is the dynamic viewpoint on quasi-isometry by Gromov [Gro93, 0.2.C₂']. Accordingly the existence of a topological coupling of groups Λ, Γ (see theorem 2.2) is a characterizing property of Λ, Γ being quasi-isometric. In the measurable setting, it is well known that a measure equivalence of groups gives rise to a weak orbit equivalence. In our situation we obtain a sort of topological version of weak orbit equivalence (see subsection 2.2) which induces an isomorphism between certain transformation groupoids of Λ and Γ (lemma 2.10). To such a transformation groupoid of a group we associate a ring (see subsection 2.4) containing the group ring up to restriction to an idempotent. It is crucial that we have a good control over the passage from the group ring of Λ resp. Γ to its groupoid ring (see e.g. lemma 2.19), and that the groupoid rings of Λ, Γ are isomorphic. Hence we can compare Λ and Γ algebraically.

2.1. Topological and couplings. Let us recall the standard definition of a quasi-isometry between finitely generated groups.

Definition 2.1. Let Γ, Λ be groups generated by the finite symmetric sets S_Γ, S_Λ and equipped with the corresponding word metrics d_Γ, d_Λ on Γ, Λ . A map $\phi : \Lambda \rightarrow \Gamma$ is called a **quasi-isometric embedding** if there are constants $\alpha \geq 1, C \geq 0$, such that for all $\lambda_1, \lambda_2 \in \Lambda$ one has

$$\alpha^{-1}d_\Lambda(\lambda_1, \lambda_2) - C \leq d_\Gamma(\phi(\lambda_1), \phi(\lambda_2)) \leq \alpha d_\Lambda(\lambda_1, \lambda_2) + C.$$

If, in addition, any $\gamma \in \Gamma$ lies within distance $\leq D$, for a constant $D \geq 0$, from the image $\phi(\Lambda)$, then ϕ is called a **quasi-isometry**. The groups Γ, Λ are called quasi-isometric if there exists a quasi-isometry $\phi : \Lambda \rightarrow \Gamma$.

The following theorem is essentially Gromov's dynamic criterion. See [Gro93, 0.2.C₂'] and especially [Sha, theorem 2.1.2].

Theorem 2.2. *For countable groups Λ, Γ consider the following statements.*

- (i) *There exists a uniform embedding $\phi : \Lambda \rightarrow \Gamma$.*
- (ii) *There exists a locally compact space X on which both Λ and Γ act continuously, freely and properly such that the two actions commute and the Γ -action is cocompact. Further, there exists fundamental domains X_Λ and X_Γ such that X_Γ is compact-open and X_Λ is closed-open. The space X is called a **topological coupling** of Λ and Γ .*
- (iii) *There exists ϕ as in (i) and a finite subset $C \subset \Gamma$ such that $\phi(\Lambda) \cdot C = \Gamma$.*
- (iv) *There exists X as in (ii) but with X_Λ being compact, i.e. the Λ -action is also cocompact.*

Then (i) is equivalent to (ii) and (iii) is equivalent to (iv). Furthermore, if Λ, Γ are finitely generated, any ϕ as in (iii) is a quasi-isometry of Λ with Γ and (iii) or (iv) are equivalent to Λ and Γ being quasi-isometric in the sense of 2.1.

Remark 2.3. This is the formulation in [Sha, theorem 2.1.2] except that there the freeness of the actions in (ii) is not demanded. Instead, it is shown that if one replaces Γ with a direct product $\Gamma \times F$ for some finite group F , then a space X can be found with the same properties as (ii) and the additional property that $X_\Gamma \subset X_\Lambda$. For us it will be important to have the version above since we want to prove theorems about the cohomological dimension cd_R over an arbitrary ring R , so we are not allowed to replace Γ by the product with a finite group without changing cd_R .

Proof. We only indicate how to modify the construction in [Sha, proof of 2.1.2] so that one gets a topological coupling as above. Let $\phi : \Lambda \rightarrow \Gamma$ be a uniform embedding. Then there is a finite subset $Q \subset \Lambda$ such that if $\phi(\lambda_1) = \phi(\lambda_2)$ then $\lambda_2^{-1}\lambda_1 \in Q$. If F is a finite group having more elements than Q , then it is easy to see that there is an injective uniform embedding $\phi' : \Lambda \rightarrow \Gamma \times F$ such that $p_\Gamma \circ \phi' = \phi$ where $p_\Gamma : \Gamma \times F \rightarrow \Gamma$ is the projection onto Γ . Choose left invariant proper (i.e. balls are finite) metrics $d_\Lambda, d_{\Gamma \times F}$ on $\Lambda, \Gamma \times F$. Then define

$$\begin{aligned} F_1(t) &= \inf\{d_{\Gamma \times F}(\phi'(\lambda_1), \phi'(\lambda_2)); d_\Lambda(\lambda_1, \lambda_2) \geq t\} \\ F_2(t) &= \sup\{d_{\Gamma \times F}(\phi'(\lambda_1), \phi'(\lambda_2)); d_\Lambda(\lambda_1, \lambda_2) \leq t\} \end{aligned}$$

Now consider the space X of all *injective* maps $\psi : \Lambda \rightarrow \Gamma \times F$ satisfying the same uniform estimate as ϕ' does:

$$(2.1) \quad F_1(d_\Lambda(\lambda_1, \lambda_2)) \leq d_{\Gamma \times F}(\psi(\lambda_1), \psi(\lambda_2)) \leq F_2(d_\Lambda(\lambda_1, \lambda_2))$$

We equip X with the pointwise-convergence topology. We let Γ act on $\Gamma \times F$ by $\gamma(\gamma', m) = (\gamma\gamma', m)$. Thereby we obtain a right Λ - and left Γ -action on X by $(\lambda\psi)(x) = \psi(\lambda x)$ and $(\gamma\psi)(x) = \gamma\psi(x)$ for $\psi \in X$. The Γ -action is free and a compact-open Γ -fundamental domain is given by $X_\Gamma = \{\psi \in X; \psi(e) \in \{e\} \times F\}$. The Γ - and Λ -action are free and proper but the latter is not cocompact in general. However, if there is a finite $C \subset \Gamma$ with

$\phi(\Lambda) \cdot C = \Gamma$, then we add the condition that all maps ψ in X satisfy $(p_\Gamma \circ \psi)(\Lambda) \cdot C = \Gamma$. Then the compact subset

$$K = \{\psi \in X; \psi(e) \in C^{-1} \times F\}$$

satisfies $K\Lambda = X$. Hence Λ acts cocompactly in this case. As for the existence of a closed-open Λ -fundamental domain, choose an enumeration $\alpha_0 = e, \alpha_1, \alpha_2, \dots$ of $\Gamma \times F$. Then define

$$E_i = \{\psi \in X; \psi(e) = \alpha_i\}$$

and $K_i = \Lambda \cdot E_i$. Then a Λ -fundamental domain is given by

$$X_\Lambda = E_0 \cup \bigcup_{i=1}^{\infty} E_i \cap K_{i-1}^C \cap \dots \cap K_0^C,$$

where $K_i^C = X - K_i$. So, if $\psi \in X_\Lambda$ and n is the minimal integer such that ψ takes the value α_n then $\psi(e) = \alpha_n$. Note that we do not necessarily have $X_\Gamma \subset X_\Lambda$. The proof that X_Λ is closed-open and the proof of all the other topological properties of X, X_Λ, X_Γ are exactly the same as in [Sha, proof of 2.1.2]. \square

Crucial for applications to amenable group is the following fact (see [Sha, theorem 2.1.7]).

Theorem 2.4. *Let X be a topological coupling of quasi-isometric groups Γ, Λ with fundamental domains X_Γ, X_Λ as in 2.2. If Γ, Λ are amenable, we can equip X with a non-trivial, ergodic $\Gamma \times \Lambda$ -invariant Borel measure such that X_Γ and X_Λ have finite measure.*

2.2. A kind of topological orbit equivalence. Let X be a topological coupling of Γ, Λ as in 2.2. Since the actions on X commute we obtain a **right Λ -action** on $\Gamma \backslash X \cong X_\Gamma$ and a **left Γ -action** on $X/\Lambda \cong X_\Lambda$. Note that we have natural homeomorphisms $X_\Lambda \times \Lambda \cong X$ and $X_\Gamma \times \Gamma \cong X$ since X_Λ and X_Γ are closed and open. To avoid confusion, we use the dot-notation " $\gamma \cdot x$ " only for the actions on the fundamental domains. We get a left Λ -action on X_Γ simply by $\lambda \cdot x = x \cdot \lambda^{-1}$, which we often use instead of the right action for symmetry reasons.

We adopt and recall the cocycle notation from [Sha, 2.2]. We define maps $\alpha : \Gamma \times X \rightarrow \Lambda$, $\beta : X \times \Lambda \rightarrow \Gamma$ by

$$(2.2) \quad \alpha(\gamma, x) = \lambda \iff (\gamma^{-1}x)\lambda \in X_\Lambda$$

$$(2.3) \quad \beta(x, \lambda) = \gamma \iff \gamma^{-1}(x\lambda) \in X_\Gamma.$$

Note that α, β are well defined because X_Λ, X_Γ are fundamental domains. The natural Γ - and Λ -actions on the fundamental domains obtained from the identification as quotients of X take the following forms.

$$(2.4) \quad \gamma \cdot x = \gamma x \alpha(\gamma^{-1}, x), \quad x \in X_\Lambda$$

$$(2.5) \quad x \cdot \lambda = \beta(x, \lambda)^{-1} x \lambda, \quad x \in X_\Gamma$$

The maps α, β satisfy the following cocycle identities on the fundamental domains.

$$(2.6) \quad \alpha(\gamma_1 \gamma_2, x) = \alpha(\gamma_1, x) \alpha(\gamma_2, \gamma_1^{-1} \cdot x) \quad \forall \gamma_1, \gamma_2 \in \Gamma, x \in X_\Lambda$$

$$(2.7) \quad \beta(x, \lambda_1 \lambda_2) = \beta(x, \lambda_1) \beta(x \cdot \lambda_1, \lambda_2) \quad \forall \lambda_1, \lambda_2 \in \Lambda, x \in X_\Gamma$$

Definition 2.5. Note that the collection of subsets of a topological space, which are open and closed, forms a set algebra. We call a map $f : X \rightarrow Y$ between topological spaces **cut-and-paste continuous** if f is measurable with respect to the set algebras of closed-open subsets.

The following lemma is a topological version of the fact from ergodic theory that a measure equivalence gives rise to a weak orbit equivalence. Unfortunately, we need the explicit formulation below in lemma 2.17, which accounts for some cumbersome cocycle calculations.

Lemma 2.6. *Let X be a topological coupling of Λ, Γ as in theorem 2.2 (ii) with closed-open fundamental domains X_Λ, X_Γ where X_Γ is compact. Let $p : X \rightarrow X_\Gamma = \Gamma \backslash X$ and $q : X \rightarrow X_\Lambda = X / \Lambda$ be the canonical projections. Then there are compact-open subsets $A \subset X_\Gamma, B \subset X_\Lambda$ that meet every orbit of the action of Λ on X_Γ resp. Γ on X_Λ such that the restriction f of q to $A \subset X_\Gamma$ is a cut-and-paste continuous bijection $f : A \rightarrow B$ satisfying the identities*

$$(2.8) \quad f(\lambda \cdot a) = \beta(a, \lambda^{-1})^{-1} \cdot f(a)$$

$$(2.9) \quad f^{-1}(\gamma \cdot b) = \left(\alpha(1, f^{-1}(b)) \alpha(\gamma^{-1}, b) \alpha(1, f^{-1}(\gamma \cdot b))^{-1} \right)^{-1} \cdot f^{-1}(b)$$

for $a \in A \cap \lambda^{-1} \cdot A, b \in B \cap \gamma^{-1} \cdot B$. Furthermore, if X carries a $\Gamma \times \Lambda$ -invariant Borel measure μ , then f is measure-preserving with respect to the restrictions $\mu|_A, \mu|_B$.

Proof. The image $B = q(X_\Gamma)$ is clearly compact and open. Note that under the identification $X = X_\Lambda \times \Lambda$ there is a finite subset $F \subset \Lambda$ such that $X_\Gamma \subset X_\Lambda \times F$, in other words, X_Γ is covered by finitely many Λ -translates of X_Λ since X_Γ is compact and X_Λ is open. Now it is easy to see that there is a closed-open subset $A \subset X_\Gamma$ such that the restriction of the projection $f = q|_A : A \rightarrow B$ is a bijection and $q(A) = q(X_\Gamma)$ holds. For an invariant measure μ on X the map f is measure-preserving since it is given by "cutting" A into finitely many pieces and translating them by elements of $F \subset \Lambda$. From $X_\Lambda \subset \Gamma X_\Gamma$ and the Γ -equivariance of q follow that $\Gamma \cdot B = X_\Lambda$, so B meets every Γ -orbit. Further, $\Lambda \cdot A = X_\Gamma$ is obtained from $q(A) = q(X_\Gamma)$.

Concerning the properties (2.8), (2.9), we can conclude from q being Γ -equivariant that

$$f(\lambda \cdot a) = f(a \cdot \lambda^{-1}) = q(\beta(a, \lambda^{-1})^{-1} a \lambda^{-1}) = \beta(a, \lambda^{-1})^{-1} \cdot q(a \lambda^{-1}) = \beta(a, \lambda^{-1})^{-1} \cdot f(a).$$

For the corresponding property of f^{-1} , consider $b \in B$ with $\gamma \cdot b \in B$. So there are $x, w \in A$ with $f(x) = b, f(w) = \gamma \cdot f(x) = \gamma \cdot b$. Equation (2.9) would follow from

$$w = \left(\alpha(1, x) \alpha(\gamma^{-1}, f(x)) \alpha(1, w)^{-1} \right)^{-1} x.$$

Note that we have $f(x) = x\alpha(1, x)$ and $f(w) = w\alpha(1, w)$ from which we obtain

$$\begin{aligned} w &= f(w)\alpha(1, w)^{-1} = (\gamma \cdot f(x))\alpha(1, w)^{-1} = \gamma f(x)\alpha(\gamma^{-1}, f(x))\alpha(1, w)^{-1} \\ &= \gamma x\alpha(1, x)\alpha(\gamma^{-1}, f(x))\alpha(1, w)^{-1} \\ &= x \cdot (\alpha(1, x)\alpha(\gamma^{-1}, f(x))\alpha(1, w)^{-1}) \end{aligned}$$

as desired. The last equality follows from the identities (2.3), (2.5). \square

2.3. Transformation groupoids.

Definition 2.7. Let Y be a topological space. Define $PI(Y)$ to be the set of partial bijections f of Y from a closed-open subset $\text{dom}(f)$ onto a closed-open subset $\text{ran}(f)$ such that f, f^{-1} are cut-and-paste continuous. A **cut-and-paste continuous pseudo-action** of a group Γ on X is a map $\sigma : \Gamma \rightarrow PI(Y)$ such that $\sigma(1_\Gamma) = \text{id}_Y$ and $\sigma(\gamma_1\gamma_2^{-1}) = \sigma(\gamma_1) \circ \sigma(\gamma_2)^{-1}$ hold, where \circ is the composition of partial maps.

Remark 2.8. The only example of a pseudo-action we will consider arises as the restriction of an ordinary continuous group action of a group Γ on a topological space Y to an open-closed subset $A \subset Y$. This pseudo-action is explicitly defined by $\sigma : \Gamma \rightarrow PI(A)$ sending $\gamma \in \Gamma$ to $\sigma(\gamma) : A \cap \gamma^{-1}A \rightarrow A \cap \gamma A, y \mapsto \gamma y$.

Definition 2.9. Let Y be a topological space equipped with a cut-and-paste continuous pseudo-action $\sigma : \Gamma \rightarrow PI(Y)$. Then we define the **transformation groupoid** $Y \rtimes \Gamma$ as

$$Y \rtimes \Gamma = \{(y, \gamma); y \in \text{dom}(\sigma(\gamma)), \gamma \in \Gamma\}.$$

The unit space of this groupoid is given by Y , the source and target maps are $s(y, \gamma) = y$, $t(y, \gamma) = \sigma(\gamma)(y) = \gamma y$ and the product and the inverse are defined by

$$\begin{aligned} (x, \gamma)(y, \gamma') &= (x, \gamma'\gamma) \\ (y, \gamma)^{-1} &= (\gamma y, \gamma^{-1}). \end{aligned}$$

Lemma 2.10. We retain the notation of lemma 2.6. The map

$$\phi : A \rtimes \Lambda \longrightarrow B \rtimes \Gamma, (a, \lambda) \mapsto (f(a), \beta(a, \lambda^{-1})^{-1})$$

is an isomorphism of groupoids. Its inverse is explicitly given by

$$\phi^{-1}(b, \gamma) = \left(f^{-1}(b), \left(\alpha(1, f^{-1}(b))\alpha(\gamma^{-1}, b)\alpha(1, f^{-1}(\gamma \cdot b))^{-1} \right)^{-1} \right).$$

Proof. Lemma 2.6 yields that ϕ is well defined, i.e. $f(a) \in \text{dom}(\sigma(\beta(a, \lambda^{-1})^{-1}))$ since $\beta(a, \lambda^{-1})^{-1} \cdot f(a) = f(\lambda \cdot a) \in B$. Further, ϕ is a groupoid morphism because of the cocycle identity (2.7):

$$\begin{aligned} \phi((a, \lambda)(\lambda \cdot a, \lambda')) &= \phi((a, \lambda'\lambda)) \\ &= (f(a), \beta(a, \lambda^{-1}\lambda'^{-1})^{-1}) \\ &= (f(a), \beta(\lambda \cdot a, \lambda'^{-1})^{-1}\beta(a, \lambda^{-1})^{-1}) \\ &= \phi(a, \lambda)\phi(\lambda \cdot a, \lambda') \end{aligned}$$

Next we only show that $\phi^{-1} \circ \phi = \text{id}$ as $\phi \circ \phi^{-1} = \text{id}$ follows along the same lines. For that we have to check the following cocycle identity

$$(2.10) \quad \alpha(1, a)\alpha(\beta(a, \lambda^{-1}), b)\alpha(1, \lambda \cdot a)^{-1} = \lambda^{-1}$$

for $a \in A$ and $b = f(a)$. From (2.2) and (2.5) we obtain

$$\beta(a, \lambda^{-1})^{-1} a \lambda^{-1} \alpha(1, \lambda \cdot a) \in X_\Lambda.$$

Note that $b = f(a) = a\alpha(1, a) \in X_\Lambda$. So we obtain from (2.2) that

$$\beta(a, \lambda^{-1})^{-1} a \alpha(1, a) \alpha(\beta(a, \lambda^{-1}), b) = \beta(a, \lambda^{-1})^{-1} b \alpha(\beta(a, \lambda^{-1}), b) \in X_\Lambda.$$

The two preceding equations and the fact that X_Λ is a Λ -fundamental domain together imply (2.10). \square

2.4. Some algebraic objects associated to group actions.

Definition 2.11. Let R be a ring, and be Y a topological space equipped with a cut-and-paste continuous pseudo-action σ of Γ . The **algebraic groupoid ring** $R(Y \rtimes \Gamma)$ of the transformation groupoid $Y \rtimes \Gamma$ is the set of all functions $f : Y \rtimes \Gamma \rightarrow R$ with the following properties.

- (i) For any $\gamma \in \Gamma$, the map $y \mapsto f(y, \gamma)$ from $\text{dom}(\sigma(\gamma))$ to R is cut-and-paste continuous in the sense that the preimages of all $r \in R$ are open and closed.
- (ii) There is a finite subset $F \subset \Gamma$ such that $f(y, \gamma) = 0$ if $\gamma \notin F$.

Then $R(Y \rtimes \Gamma)$ becomes a ring by pointwise addition and the convolution product

$$(fg)(x, \gamma) = \sum_{(x, \gamma')(y, \gamma'')=(x, \gamma)} f(y, \gamma'')g(x, \gamma').$$

Note that (ii) ensures that the sum is finite. Now assume that Y is equipped with a finite Borel measure μ which is invariant under the pseudo-action of Γ . Then $\mathbb{C}(Y \rtimes \Gamma)$ comes equipped with the normalized **trace**

$$\text{tr}(f) = \frac{1}{\mu(Y)} \int_{Y \times \{1_\Gamma\}} f d\mu.$$

The notion *trace* is justified by the following lemma.

Lemma 2.12. *The functional $\text{tr} : \mathbb{C}(Y \rtimes \Gamma) \rightarrow \mathbb{C}$ satisfies the trace property, i.e. $\text{tr}(fg) = \text{tr}(gf)$ for $f, g \in \mathbb{C}(Y \rtimes \Gamma)$.*

Proof. We have

$$\text{tr}(fg) = \sum_{\gamma \in \Gamma} \int_{\text{dom}(\sigma(\gamma))} f(\gamma x, \gamma^{-1}) g(x, \gamma) d\mu(x).$$

Because of the invariance of μ this equals

$$\begin{aligned} \sum_{\gamma \in \Gamma} \int_{\text{dom}(\sigma(\gamma^{-1}))} f(y, \gamma^{-1}) g(\gamma^{-1}y, \gamma) d\mu(y) &= \sum_{\gamma \in \Gamma} \int_{\text{dom}(\sigma(\gamma))} g(\gamma y, \gamma^{-1}) f(y, \gamma) d\mu(y) \\ &= \text{tr}(gf). \end{aligned}$$

\square

Definition 2.13. The ring of cut-and-paste continuous functions $Y \rightarrow R$ is denoted by $\mathcal{F}(Y; R)$, which is the same as $R(Y \rtimes 1)$ in the notation above. If Γ is acting continuously on Y and hence on $\mathcal{F}(Y; R)$ by $f^\gamma(y) = f(\gamma^{-1}y)$, one can define the **crossed product ring** $\mathcal{F}(Y; R) \rtimes \Gamma$ as follows. As an abelian group $\mathcal{F}(Y; R) \rtimes \Gamma$ is the free $\mathcal{F}(Y; R)$ -module $\mathcal{F}(Y; R)[\Gamma]$ with basis Γ . Its multiplication is determined by the rule $f\gamma g\gamma' = fg^\gamma\gamma\gamma'$ for $f, g \in \mathcal{F}(Y; R)$ and $\gamma, \gamma' \in \Gamma$.

Obviously, $\mathcal{F}(Y; R)$ is a subring of $R(Y \rtimes \Gamma)$, and it is clear from property (ii) in the definition of the algebraic groupoid ring that $R(Y \rtimes \Gamma)$ is a free $\mathcal{F}(Y; R)$ -module with basis Γ – like the crossed product ring. Indeed, the map below identifies the two rings.

Remark 2.14. The map defined by

$$\mathcal{F}(Y; R) \rtimes \Gamma \longrightarrow R(Y \rtimes \Gamma), \quad \sum_{\gamma} f_{\gamma} \gamma \mapsto ((y, \gamma) \mapsto f_{\gamma}(\gamma y))$$

is a natural ring isomorphism. Moreover, if Y carries a finite Γ -invariant Borel measure μ , then $\text{tr}(\sum_{\gamma} f_{\gamma} \gamma) = \mu(Y)^{-1} \int_Y f_1 d\mu$ defines a trace on $\mathcal{F}(Y; R) \rtimes \Gamma$, and the isomorphism above clearly preserves the traces. The crossed product has the advantage of being a universal construction, whereas theorem 2.17 is more easily seen with the groupoid ring. In the sequel we *identify these two rings*.

Remark 2.15. Via the natural isomorphism $\mathcal{F}(Y; R) \cong \mathcal{F}(Y; R) \rtimes \Gamma \otimes_{R\Gamma} R$ the ring of functions $\mathcal{F}(Y; R)$ becomes an $\mathcal{F}(Y; R) \rtimes \Gamma$ -module. Explicitly, this module structure is given as follows. Let $f, g \in \mathcal{F}(Y; R)$ and $\gamma \in \Gamma$. Then we have $(f\gamma) \cdot g = fg^\gamma$.

Remark 2.16. For any ring R and an idempotent $p \in R$, i.e. $p^2 = p$, the set pRp is again a ring with unit p . The characteristic function of a closed-open subset $A \subset Y$, usually denoted by χ_A , is an idempotent in $\mathcal{F}(Y; R)$. The action of Γ on Y restricts to a pseudo-action on A (see remark 2.8). Then there is an obvious identification

$$\chi_A \mathcal{F}(Y; R) \rtimes \Gamma \chi_A = \chi_A R(Y \rtimes \Gamma) \chi_A = R(A \rtimes \Gamma).$$

Lemma 2.17. We retain the notation of 2.10 and of 2.6. The map $\phi : A \rtimes \Lambda \longrightarrow B \rtimes \Gamma$ induces a ring isomorphism

$$\Phi : R(B \rtimes \Gamma) \xrightarrow{\cong} R(A \rtimes \Lambda), \quad g \mapsto g \circ \phi.$$

Now assume that the topological coupling X is equipped with a non-trivial $\Gamma \times \Lambda$ -invariant Borel measure μ such that X_Γ, X_Λ have finite measure, and let $R = \mathbb{C}$. Then the induced Γ -resp. Λ -action on X_Λ resp. X_Γ is μ -preserving and Φ preserves the (normalized) traces.

Proof. Since the fundamental domains X_Γ, X_Λ are closed and open and the actions are continuous, the sets $\{x; \alpha(\gamma, x) = \lambda\}, \{x; \beta(x, \lambda) = \gamma\}$ are closed and open for fixed $\gamma \in \Gamma, \lambda \in \Lambda$. It follows from the explicit formula in lemma 2.10 that $g \circ \phi$ is cut-and-paste continuous for $g \in R(B \rtimes \Gamma)$. Similarly, $h \circ \phi^{-1}$ is cut-and-paste continuous for $h \in R(A \rtimes \Lambda)$. For any compact $K \subset X$ and fixed $\gamma \in \Gamma, \lambda \in \Lambda$ the (restrictions of the) cocycles $\alpha : \{\gamma\} \times K \rightarrow \Lambda, \beta : K \times \{\lambda\} \rightarrow \Gamma$ have finite image since both fundamental domains are open. Again using the explicit formulas, this implies that $g \circ \phi$ and $h \circ \phi^{-1}$ satisfy property (ii) in definition 2.11. So Φ is well defined with the map induced by ϕ^{-1} as its inverse. Finally suppose X comes equipped with a measure μ as described in

the hypothesis. Since μ is $\Gamma \times \Lambda$ -invariant it is clear from 2.5 that the induced actions on the fundamental domains are measure preserving. Now let $g \in R(B \rtimes \Gamma)$. For the following computation note that $\beta(a, 1) = 1$ for $a \in A$ and that the map $f : A \rightarrow B$ from lemma 2.6 is measure preserving, in particular $\mu(A) = \mu(B)$. Therefore we obtain

$$\begin{aligned} \text{tr}(g \circ \phi) &= \frac{1}{\mu(A)} \int_A g(f(a), \beta(a, 1)^{-1}) d\mu(a) \\ &= \frac{1}{\mu(A)} \int_A g(f(a), 1) d\mu(a) \\ &= \frac{1}{\mu(B)} \int_B g(b, 1) d\mu(b) = \text{tr}(g). \end{aligned}$$

□

Lemma 2.18. *For any ring R and a compact topological space Y the ring of cut-and-paste continuous functions $\mathcal{F}(Y; R)$ is a flat R -module.*

Proof. For a covering \mathcal{U} of Y consisting of mutually disjoint subsets that are closed and open – let’s call it a cut-and-paste covering – the R -submodule $\mathcal{F}_{\mathcal{U}}(Y; R) \subset \mathcal{F}(Y; R)$ is defined by

$$\mathcal{F}_{\mathcal{U}}(Y; R) = \{f \in \mathcal{F}(Y; R); f|_U \text{ constant for all } U \in \mathcal{U}\}.$$

Note that a cut-and-paste covering has finitely many elements because Y is compact. Obviously, $\mathcal{F}_{\mathcal{U}}(Y; R)$ is isomorphic to $\bigoplus_{\mathcal{U}} R$, in particular it is a flat R -module. Further $\mathcal{F}(Y; R)$ is the union $\bigcup_{\mathcal{U}} \mathcal{F}_{\mathcal{U}}(Y; R)$ where \mathcal{U} runs through all cut-and-paste coverings. Note that this is a directed union; the coverings form a directed set with respect to the “being finer”-relation. Since directed colimits and, in particular, directed unions of flat modules, are again flat [Lam99, prop. (4.4) on p. 123], we are done. □

From that we easily deduce the following crucial fact, which will be used throughout this paper.

Lemma 2.19. *Let R be a ring R , and be B a compact subset of a topological space Y . Then $\mathcal{F}(Y; R) \rtimes \Gamma \chi_B$ is a flat $R\Gamma$ -module.*

Proof. Just note that $M \otimes_{R\Gamma} \mathcal{F}(Y; R) \rtimes \Gamma \chi_B \cong M \otimes_R \mathcal{F}(B; R)$ and use lemma 2.18. □

Remark 2.20. We recall some basics about Morita equivalences (cf. [Lam99, (18.30) on p. 490]). Let R be a ring and be $p \in R$ an idempotent, i.e. $p^2 = p$. Then p is called **full** if $RpR = R$ holds. In this case, the rings pRp and R are **Morita equivalent** and tensoring with the bimodules pR resp. Rp provides equivalences of their respective module categories. Furthermore, these equivalences are exact and preserve projective modules.

Lemma 2.21. *Let Y be a topological space on which Γ acts continuously, and be $A \subset Y$ a compact-open subset which meets every Γ -orbit, i.e. $\Gamma A = Y$. Let P be a projective left $R(Y \rtimes \Gamma)$ -module. Then $\chi_A P$ is a projective module over $\chi_A R(Y \rtimes \Gamma) \chi_A = R(A \rtimes \Gamma)$. Furthermore, if Y is compact, then χ_A is a full idempotent in $R(Y \rtimes \Gamma)$, hence the rings $R(Y \rtimes \Gamma)$ and $R(A \rtimes \Gamma)$ are Morita equivalent.*

Proof. Recall that we identify $\mathcal{F}(Y; R) \rtimes \Gamma$ with $R(Y \rtimes \Gamma)$. For the first assertion, it suffices to treat the case $P = \mathcal{F}(Y; R) \rtimes \Gamma$. By assumption we have $Y = \bigcup_{\gamma \in \Gamma} \gamma A$. Since the open-closed sets form a set algebra, there are open-closed $A_\gamma \subset A$, $\gamma \in \Gamma$, such that $Y = \bigcup_{\gamma \in \Gamma} \gamma A_\gamma$ is a disjoint union. Note that

$$\chi_A \mathcal{F}(Y; R) \rtimes \Gamma \chi_{\gamma A_\gamma} \cong \chi_A \mathcal{F}(Y; R) \rtimes \Gamma \chi_{A_\gamma} = \chi_A \mathcal{F}(Y; R) \rtimes \Gamma \chi_A \chi_{A_\gamma}$$

is a left projective $\chi_A \mathcal{F}(Y; R) \rtimes \Gamma \chi_A$ -module. We claim that the map

$$\omega : \chi_A \mathcal{F}(Y; R) \rtimes \Gamma \longrightarrow \bigoplus_{\gamma \in \Gamma} \chi_A \mathcal{F}(Y; R) \rtimes \Gamma \chi_{\gamma A_\gamma}, x \mapsto (x \chi_{\gamma A_\gamma})_{\gamma \in \Gamma}$$

is an isomorphism of $\chi_A \mathcal{F}(Y; R) \rtimes \Gamma \chi_A$ -modules. The only thing to check is that ω is well defined: We have to show that for $x \in \chi_A \mathcal{F}(Y; R) \rtimes \Gamma$ there is a finite subset $F(x) \subset \Gamma$ such that $x \chi_{\gamma A_\gamma} = 0$ for $\gamma \notin F(x)$. Consider $x = \sum_{i=1}^n f_i \gamma_i$ with $f_i = f_i \chi_A$. Define $F(x) = \{\gamma \in \Gamma; \exists i = 1, \dots, n : A \cap \gamma_i \gamma A_\gamma \neq \emptyset\}$. The set $F(x)$ is finite since for any fixed $i \in \{1, \dots, n\}$ the compact set A is covered by the family of disjoint open sets $\gamma_i \gamma A_\gamma$, $\gamma \in \Gamma$. So we obtain for $\gamma \notin F(x)$

$$\sum_{i=1}^n f_i \gamma_i \chi_{\gamma A_\gamma} = \sum_{i=1}^n f_i \chi_A \gamma_i \chi_{\gamma A_\gamma} = \sum_{i=1}^n f_i \chi_{A \cap \gamma_i \gamma A_\gamma} \gamma_i = 0.$$

If, additionally, Y is compact, there is a finite subset $F \subset \Gamma$ with $Y = \bigcup_{\gamma \in F} \gamma A_\gamma$, $A_\gamma \subset A$ as above. But then we get $1 = \chi_Y = \sum_{\gamma \in F} \gamma \chi_A \chi_{A_\gamma} \gamma^{-1}$, so χ_A is full. \square

3. THE INDUCTION MAP IN GROUP (CO)HOMOLOGY

Throughout this section let $\phi : \Lambda \rightarrow \Gamma$ be a **uniform embedding** (definition 1.1). By theorem 2.2 there is a topological coupling X on which Γ and Λ act in a commuting way, and closed-open fundamental domains X_Γ, X_Λ with X_Γ **being compact**. If ϕ is a quasi-isometry, then such an X exists with both X_Λ, X_Γ being compact. Further, there are compact-open subsets $A \subset X_\Gamma, B \subset X_\Lambda$ and a groupoid isomorphism $\Phi : A \rtimes \Lambda \xrightarrow{\cong} B \rtimes \Gamma$ (lemma 2.10) inducing a ring isomorphism $\Phi : R(B \rtimes \Gamma) \xrightarrow{\cong} R(A \rtimes \Lambda)$ (lemma 2.17), where R is an arbitrary commutative ring. **We fix a choice of $X, X_\Gamma, X_\Lambda, A, B$ and Φ for the rest of this section.** Recall the identification $\chi_A R(X_\Gamma \rtimes \Lambda) \chi_A = R(A \rtimes \Lambda)$ (remark 2.16). Further it will be crucial in the sequel because of its algebraic consequences (see 2.20) that the characteristic function χ_A is a full idempotent in $R(X_\Gamma \rtimes \Lambda)$ by 2.21. If X_Λ is also compact, i.e. in the case of a quasi-isometry, χ_B is also a full idempotent in $R(X_\Lambda \rtimes \Gamma)$.

Next we define an induction map from the group homology resp. cohomology of Λ to that of Γ . This induction map *depends on the topological coupling and on Φ* , but we suppress that in the notation to simplify it. The reader should be warned that because of these choices it is *not* functorial with respect to compositions of uniform embeddings. First we define induction functors on the level of modules. If $f : R \rightarrow S$ is a ring homomorphism and M an S -module, we denote by $\text{res}_f M$ the R -module M induced by f . If

f is the inclusion of a subring $R \subset S$ we also write $\text{res}_R M$. Consider the functors

$$(3.1) \quad \underline{I} : \{R\Lambda\text{-modules}\} \longrightarrow \{R\Gamma\text{-modules}\}$$

$$M \longmapsto \underline{I}(M) = \text{res}_{R\Gamma} \left(R(X_\Lambda \rtimes \Gamma) \chi_B \otimes_{R(B \rtimes \Gamma)} \text{res}_\Phi (\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M) \right)$$

and

$$(3.2) \quad \bar{I} : \{R\Lambda\text{-modules}\} \longrightarrow \{R\Gamma\text{-modules}\}$$

$$M \longmapsto \bar{I}(M) = \text{res}_{R\Gamma} \left(\text{hom}_{R(B \rtimes \Gamma)} (\chi_B R(X_\Lambda \rtimes \Gamma), \text{res}_\Phi (\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M)) \right)$$

Explicitly, the (left) $R\Gamma$ -module structure on $\bar{I}(M)$ is given by $(\gamma f)(x) = f(x\gamma)$, $x \in \chi_B R(X_\Lambda \rtimes \Gamma)$.

Lemma 3.1. *Assume that $\phi : \Lambda \rightarrow \Gamma$ is a quasi-isometry, in particular X_Λ is compact. Then $\bar{I}(R)$ and $\underline{I}(R)$ are both isomorphic to $\mathcal{F}(X_\Lambda) = \mathcal{F}(X_\Lambda; R)$ as $R\Gamma$ -modules.*

Proof. For $\underline{I}(R)$ this follows from

$$\begin{aligned} \underline{I}(R) &= \text{res}_{R\Gamma} \left(R(X_\Lambda \rtimes \Gamma) \chi_B \otimes_{R(B \rtimes \Gamma)} \text{res}_\Phi (\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} R) \right) \\ &= \text{res}_{R\Gamma} \left(R(X_\Lambda \rtimes \Gamma) \chi_B \otimes_{R(B \rtimes \Gamma)} \text{res}_\Phi \mathcal{F}(A) \right) \\ &= \text{res}_{R\Gamma} \left(R(X_\Lambda \rtimes \Gamma) \chi_B \otimes_{R(B \rtimes \Gamma)} \mathcal{F}(B) \right) \\ &= \text{res}_{R\Gamma} \left(R(X_\Lambda \rtimes \Gamma) \otimes_{R(X_\Lambda \rtimes \Gamma)} \mathcal{F}(X_\Lambda) \right) \\ &= \text{res}_{R\Gamma} \mathcal{F}(X_\Lambda). \end{aligned}$$

The fourth equality comes from the fact that χ_B is a full idempotent in $R(X_\Lambda \rtimes \Gamma)$ if X_Λ is compact. Similarly, one concludes $\bar{I}(R) \cong \mathcal{F}(X_\Lambda)$. \square

Theorem 3.2 (Induction for Uniform Embeddings). *Let M be an $R\Lambda$ -module. Then there are homomorphisms, natural in M ,*

$$I_n : H_n(\Lambda, M) \longrightarrow H_n(\Gamma, \underline{I}(M)), \quad n \geq 0,$$

*between the group homology of Λ and Γ with coefficients in M resp. $\underline{I}(M)$. We say I_n is the **induction homomorphism** (with respect to the topological coupling X and the groupoid map Φ above). The map I_n factorizes as*

$$H_n(\Lambda, M) \longrightarrow H_n(\Lambda, \text{res}_{R\Lambda} R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M) \xrightarrow{\cong} H_n(\Gamma, \underline{I}(M)),$$

where the first map is induced by the map of the coefficients $j : M \rightarrow R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M, m \mapsto 1 \otimes m$, and the second map is an isomorphism. Similarly, there are natural homomorphisms

$$I^n : H^n(\Lambda, M) \longrightarrow H^n(\Gamma, \bar{I}(M)), \quad n \geq 0,$$

factorizing as $H^n(\Lambda, M) \longrightarrow H^n(\Lambda, \text{res}_{R\Lambda} R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M) \xrightarrow{\cong} H^n(\Gamma, \bar{I}(M))$, where again the first map is induced by j and the second map is an isomorphism. Furthermore, if ϕ is

an quasi-isometry, in particular X_Λ is compact, then the induction homomorphisms for $M = R$

$$\begin{aligned} H_*(\Lambda, R) &\rightarrow H_*(\Gamma, \mathcal{F}(X_\Lambda)) \\ H^*(\Lambda, R) &\rightarrow H^*(\Gamma, \mathcal{F}(X_\Lambda)) \end{aligned}$$

are compatible with cup- and cap-products.

Proof. The proof occupies the rest of this section. We prefer to be slightly redundant in separating the argument for the compatibility of the product structures from the rest, thereby improving the readability.

Let P_* be a right projective $R\Lambda$ -resolution of the trivial module R , and be Q_* a right projective $R\Gamma$ -resolution of R . By lemma 2.19 $R(X_\Gamma \rtimes \Lambda)$ is flat over $R\Lambda$ and $R(X_\Lambda \rtimes \Gamma)\chi_B$ is flat over $R\Gamma$. In particular, $P_* \otimes_{R\Lambda} R(X_\Gamma \rtimes \Lambda)$ and $Q_* \otimes_{R\Gamma} R(X_\Lambda \rtimes \Gamma)\chi_B$ are $R(X_\Gamma \rtimes \Lambda)$ -resp. $R(B \rtimes \Gamma)$ -resolutions of $\mathcal{F}(X_\Gamma; R)$ resp. $\mathcal{F}(B; R)$. Both are projective resolutions; for the first this obvious, for the latter this is due to lemma 2.21. In particular, if M, N are left resp. right $R(X_\Gamma \rtimes \Lambda)$ -modules, then we have a canonical isomorphism $N \otimes_{R(X_\Gamma \rtimes \Lambda)} M \cong N\chi_A \otimes_{R(A \rtimes \Lambda)} \chi_A M$ since χ_A is a full idempotent. Now the composition of the following chain maps induces $I_n, n \geq 0$, in homology.

$$\begin{array}{c} P_* \otimes_{R\Lambda} M \\ \downarrow 1 \\ P_* \otimes_{R\Lambda} (R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M) \\ \downarrow 2 \\ (P_* \otimes_{R\Lambda} R(X_\Gamma \rtimes \Lambda)) \otimes_{R(X_\Gamma \rtimes \Lambda)} (R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M) \\ \downarrow 3 \\ (P_* \otimes_{R\Lambda} R(X_\Gamma \rtimes \Lambda)\chi_A) \otimes_{R(A \rtimes \Lambda)} (\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M) \\ \downarrow 4 \\ \text{res}_\Phi \left((P_* \otimes_{R\Lambda} R(X_\Gamma \rtimes \Lambda)\chi_A) \right) \otimes_{R(B \rtimes \Gamma)} \text{res}_\Phi \left(\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M \right) \\ \downarrow 5 \\ \underbrace{Q_* \otimes_{R\Gamma} R(X_\Lambda \rtimes \Gamma)\chi_B \otimes_{R(B \rtimes \Gamma)} \text{res}_\Phi \left(\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M \right)}_{=I(M)} \end{array}$$

The first map is coming from the inclusion $M \rightarrow R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M, m \mapsto 1 \otimes m$. The second map and the third map are the canonical identifications, and the fourth map is the obvious isomorphism coming from the ring isomorphism $\Phi : R(B \rtimes \Gamma) \xrightarrow{\cong} R(A \rtimes \Lambda)$. Note that $\text{res}_\Phi(P_* \otimes_{R\Lambda} R(X_\Gamma \rtimes \Lambda)\chi_A)$ and $Q_* \otimes_{R\Gamma} R(X_\Lambda \rtimes \Gamma)\chi_B$ are projective resolutions of the same module $\mathcal{F}(B; R)$. Now the fifth map is the homotopy equivalence we get from the fundamental theorem of homological algebra, which is

unique up to chain homotopy.

Now let us turn to the cohomological case. Let P_* , Q_* be left projective $R\Lambda$ - resp. $R\Gamma$ -resolutions of R . We only describe the dual diagram:

$$\begin{array}{c}
\mathrm{hom}_{R\Lambda}(P_*, M) \\
\downarrow 1 \\
\mathrm{hom}_{R\Lambda}(P_*, R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M) \\
\downarrow 2 \\
\mathrm{hom}_{R(X_\Gamma \rtimes \Lambda)}(R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} P_*, R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M) \\
\downarrow 3 \\
\mathrm{hom}_{R(A \rtimes \Lambda)}(\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} P_*, \chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M) \\
\downarrow 4 \\
\mathrm{hom}_{R(B \rtimes \Gamma)}(\mathrm{res}_\Phi(\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} P_*), \mathrm{res}_\Phi(\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M)) \\
\downarrow 5 \\
\mathrm{hom}_{R(B \rtimes \Gamma)}(\chi_B R(X_\Lambda \rtimes \Gamma) \otimes_{R\Gamma} Q_*, \mathrm{res}_\Phi(\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M))
\end{array}$$

The five maps correspond to the maps in the homological diagram. Now the last term is *canonically* isomorphic to $\mathrm{hom}_{R\Gamma}(Q_*, \tilde{I}(M))$ by applying two canonical isomorphisms of pairs of adjoint functors, of the pair $\chi_B R(X_\Lambda \rtimes \Gamma) \otimes_{R(X_\Lambda \rtimes \Gamma)}$ - and $\mathrm{hom}_{R(B \rtimes \Gamma)}(\chi_B R(X_\Lambda \rtimes \Gamma), -)$ and of the pair $R(X_\Lambda \rtimes \Gamma) \otimes_{R\Gamma}$ - and $\mathrm{res}_{R\Gamma}(-)$:

$$\begin{aligned}
& \mathrm{hom}_{R(B \rtimes \Gamma)}(\chi_B R(X_\Lambda \rtimes \Gamma) \otimes_{R\Gamma} Q_*, \mathrm{res}_\Phi(\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M)) \\
& \cong \mathrm{hom}_{R(X_\Lambda \rtimes \Gamma)}(R(X_\Lambda \rtimes \Gamma) \otimes_{R\Gamma} Q_*, \mathrm{hom}_{R(B \rtimes \Gamma)}(\chi_B R(X_\Lambda \rtimes \Gamma), \mathrm{res}_\Phi(\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M))) \\
& \cong \mathrm{hom}_{R\Gamma}(Q_*, \underbrace{\mathrm{res}_{R\Gamma} \mathrm{hom}_{R(B \rtimes \Gamma)}(\chi_B R(X_\Lambda \rtimes \Gamma), \mathrm{res}_\Phi(\chi_A R(X_\Gamma \rtimes \Lambda) \otimes_{R\Lambda} M))}_{=\tilde{I}(M)})
\end{aligned}$$

Now we turn to the proof of the compatibility with cup- and cap-products in the case of a quasi-isometry, i.e. X_Λ is compact. In that case χ_B is a full idempotent which enables

us to write the induction map in the following way.

$$\begin{array}{c}
H^*(\Lambda, R) = \text{Ext}_{R\Lambda}^*(R, R) \\
\downarrow \text{(A.1)} \\
\text{Ext}_{R(X_\Gamma \rtimes \Lambda)}^*(\mathcal{F}(X_\Gamma), \mathcal{F}(X_\Gamma)) \\
\uparrow \cong \text{(A.5)} \\
\text{Ext}_{R(A \rtimes \Lambda)}^*(\mathcal{F}(A), \mathcal{F}(A)) \\
\downarrow \cong \\
\text{Ext}_{R(B \rtimes \Gamma)}^*(\mathcal{F}(B), \mathcal{F}(B)) \\
\downarrow \cong \text{(A.5)} \\
H^*(\Gamma, \mathcal{F}(X_\Lambda)) = \text{Ext}_{R(X_\Lambda \rtimes \Gamma)}^*(\mathcal{F}(X_\Lambda), \mathcal{F}(X_\Lambda))
\end{array}$$

At this point we need the product structures on Tor- and Ext-Groups explained in the appendix. By lemma A.1 the cup and the composition product on $H^*(\Lambda, R)$ coincide. The homomorphisms (A.1) and (A.5) defined in the appendix are compatible with the composition product. The isomorphism in the middle comes from the ring isomorphism $\Phi : R(B \rtimes \Gamma) \rightarrow R(A \rtimes \Lambda)$ and the fact that $\text{res}_\Phi \mathcal{F}(A) \cong \mathcal{F}(B)$, so it also respects the composition product. The composition and the cup product on the last term $H^*(\Gamma, \mathcal{F}(X_\Lambda))$ coincide by lemma A.2. This shows the compatibility with respect to the cup product. The induction in homology can be expressed analogously in terms of Tor and the maps (A.2) and (A.6). Since these maps are compatible with respect to the evaluation product and since the evaluation and cap product coincide on the first and last term of the composition by the same lemma, the proof is now completed. \square

4. QUASI-ISOMETRY AND (CO)HOMOLOGICAL DIMENSION

In this section we prove theorem 1.2. Recall that the **homological dimension** $\text{hd}_R(\Gamma)$ of a group Γ over a ring R is defined as

$$\text{hd}_R(\Gamma) = \sup\{n; \exists R\Gamma\text{-module } M \text{ with } H_n(\Gamma, M) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

In the same way one defines the cohomological dimension $\text{cd}_R(\Gamma)$. It is a basic fact in group homology ([Wei94, lemma 4.1.10], [Bro94, p. 185]) that $\text{hd}_R(M)$ is the minimal number n such that there is a resolution of the trivial $R\Gamma$ -module R

$$0 \leftarrow R \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n \leftarrow 0$$

by *flat* $R\Gamma$ -modules F_i . Analogously, the cohomological dimension $\text{cd}_R(\Gamma)$ is the minimal n such that there is a *projective* $R\Gamma$ -resolution of R of length n .

We prove theorem 1.2 only for the homological dimension; it is an easy matter of dualizing to prove the cohomological statement. Let us turn to statement (i). Suppose that Λ uniformly embeds into Γ , and assume $n = \text{hd}_R(\Lambda) < \infty$. Choose an $R\Lambda$ -module M with $H_n(\Lambda, M) \neq 0$ and a flat $R\Lambda$ -resolution F_i , $0 \leq i \leq n$, of length n as above. The

claim would follow from the injectivity of the induction map I_n in theorem 3.2. Write $\mathcal{F}(Y) = \mathcal{F}(Y; R)$. So it suffices to show that for any compact space Y with a continuous Λ -action the homomorphism

$$(4.1) \quad H_n(\Lambda, M) \longrightarrow H_n(\Lambda, \mathcal{F}(Y) \rtimes \Lambda \otimes_{R\Lambda} M),$$

which is induced by $M \rightarrow \mathcal{F}(Y; R) \rtimes \Lambda \otimes_{R\Lambda} M, m \mapsto 1 \otimes m$, is injective. But because of

$$H_n(\Lambda, \mathcal{F}(Y) \rtimes \Lambda \otimes_{R\Lambda} M) = \ker \left(F_n \otimes_{R\Lambda} \mathcal{F}(Y) \rtimes \Lambda \otimes_{R\Lambda} M \rightarrow F_{n-1} \otimes_{R\Lambda} \mathcal{F}(Y) \rtimes \Lambda \otimes_{R\Lambda} M \right) \\ H_n(\Lambda, M) = \ker \left(F_n \otimes_{R\Lambda} M \rightarrow F_{n-1} \otimes_{R\Lambda} M \right)$$

we are reduced to show that for a flat $R\Lambda$ -module F the map

$$\sigma_F : F \otimes_{R\Lambda} M \longrightarrow F \otimes_{R\Lambda} \mathcal{F}(Y) \rtimes \Lambda \otimes_{R\Lambda} M, x \otimes m \mapsto x \otimes 1 \otimes m$$

is injective. First note that $\sigma_{R\Lambda} : M \rightarrow R\Lambda \otimes_{R\Lambda} \mathcal{F}(Y) \rtimes \Lambda \otimes_{R\Lambda} M = \mathcal{F}(Y) \otimes_R M$ is injective since the R -linear map $R \rightarrow \mathcal{F}(Y)$ (inclusion of constant functions) is split by the evaluation $\mathcal{F}(Y) \rightarrow R, f \mapsto f(y_0)$ at some base point y_0 . Hence σ_F is injective for any free module F . By a theorem of Lazard and Govorov [Lam99, theorem (4.34) on p. 134] any flat module is the directed colimit of free modules and taking directed colimits is an exact functor [Wei94, theorem 2.6.15], hence σ_F is injective for any flat module.

Next we prove statement (ii) of theorem 1.2. Suppose that Λ is amenable and that R contains \mathbb{Q} . Then the injectivity of (4.1) is obtained as follows. Because of amenability, we can equip Y with a Λ -invariant probability measure. By composing that with a \mathbb{Q} -linear map $\mathbb{R} \rightarrow \mathbb{Q}$ that maps 1 to 1, we obtain a signed finitely additive, \mathbb{Q} -valued, Λ -invariant probability measure μ on Y . Since a function in $f \in \mathcal{F}(Y)$ takes only finitely many values, we obtain a well defined integration $\int_Y f d\mu$. This little trick is taken from [Sha, proof of theorem 1.5]. Further, the inclusion $R\Lambda \hookrightarrow \mathcal{F}(Y) \rtimes \Lambda$ is split by the $R\Lambda$ -bimodule map

$$\mathcal{F}(Y) \rtimes \Lambda \longrightarrow R\Lambda, \sum f_i \lambda_i \mapsto \sum \left(\int_Y f_i d\mu \right) \lambda_i.$$

Hence we get a $R\Lambda$ -linear left inverse of the map $M \rightarrow \mathcal{F}(Y) \rtimes \Lambda \otimes_{R\Lambda} M$, and so (4.1) is injective. This finishes the proof of theorem 1.2.

5. QUASI-ISOMETRY AND THE COHOMOLOGY RING OF A NILPOTENT GROUP

5.1. A module structure on the reduced cohomology. We briefly recall the definition of the reduced cohomology $\bar{H}^n(\Gamma, V)$ of a discrete group Γ with coefficients in a unitary or orthonormal representation V of Γ . It is defined in terms of the **standard homogeneous resolution**. Consider the chain complex

$$(5.1) \quad C^n(\Gamma, V) = \{ \omega : \Gamma^{n+1} \rightarrow V; \omega(\gamma\gamma_0, \dots, \gamma\gamma_n) = \gamma\omega(\gamma_0, \dots, \gamma_n) \}$$

equipped with the standard homogeneous differential $d^n : C^n(\Gamma, V) \rightarrow C^{n+1}(\Gamma, V)$

$$(d^n \omega)(\gamma_0, \dots, \gamma_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \omega(\gamma_0, \dots, \hat{\gamma}_i, \dots, \gamma_{n+1}).$$

Then $C^n(\Gamma, V)$ carries the topology of pointwise convergence. The space of n -cocycles $\ker d^n$ is closed with respect to this topology, but the space of boundaries $\operatorname{im} d^{n-1}$ need not be a closed subspace. The **reduced cohomology** $\overline{H}^n(\Gamma, V)$ is defined by kernel modulo the closure of the image:

$$\overline{H}^n(\Gamma, V) = \frac{\ker(d^n : C^n(\Gamma, V) \rightarrow C^{n+1}(\Gamma, V))}{\operatorname{clos}\left(\operatorname{im}(d^{n-1} : C^{n-1}(\Gamma, V) \rightarrow C^n(\Gamma, V))\right)}$$

There is an obvious surjection $H^n(\Gamma, V) \rightarrow \overline{H}^n(\Gamma, V)$.

Let Y be a compact space equipped with a Γ -invariant finite Borel measure μ , and denote by $L^2(Y; \mu)$ the μ -square-integrable real functions on Y . We write $\mathcal{F}(Y)$ for $\mathcal{F}(Y; \mathbb{R})$. Since $\mathcal{F}(Y)$ is a ring (multiplication of functions) with a Γ -equivariant multiplication, the cohomology $H^*(Y, \mathcal{F}(Y))$ is a graded ring by its cup-product (see also the appendix). In terms of the standard homogeneous resolution the product of two cocycles $f : \Gamma^{n+1} \rightarrow \mathcal{F}(Y)$, $g : \Gamma^{m+1} \rightarrow \mathcal{F}(Y)$ is explicitly given by

$$(5.2) \quad [f] \cup [g] = [\Gamma^{m+n+1} \rightarrow \mathcal{F}(Y), (\gamma_0, \dots, \gamma_{n+m}) \mapsto f(\gamma_0, \dots, \gamma_n)g(\gamma_n, \dots, \gamma_{n+m})].$$

We now exhibit a module structure on the reduced cohomology $\overline{H}^*(Y; L^2(Y; \mu))$ that turns out to be crucial for the proof of theorem 1.5. The multiplication of functions defines a Γ -equivariant map $\omega : \mathcal{F}(Y) \otimes_{\mathbb{R}} L^2(Y; \mu) \rightarrow L^2(Y; \mu)$ and hence a graded left $H^*(\Gamma, \mathcal{F}(Y))$ -module structure on $H^*(\Gamma, L^2(Y; \mu))$. In terms of the standard homogeneous resolution (5.1), this module structure is explicitly given by the same formula as above:

$$(5.3) \quad [f] \cup [g] = [\Gamma^{m+n+1} \rightarrow L^2(Y; \mu), (\gamma_0, \dots, \gamma_{n+m}) \mapsto f(\gamma_0, \dots, \gamma_n)g(\gamma_n, \dots, \gamma_{n+m})]$$

where $f : \Gamma^{n+1} \rightarrow \mathcal{F}(Y)$, $g : \Gamma^{m+1} \rightarrow L^2(Y; \mu)$ are cocycles (see [Gui80, (11.11) on p. 65]. From that we see its continuity for fixed f , and so it descends to a left $H^*(\Gamma, \mathcal{F}(Y))$ -**module structure on the reduced cohomology** $\overline{H}^*(\Gamma, L^2(Y; \mu))$. Similarly, we get a right $H^*(\Gamma, \mathcal{F}(Y))$ -module structure on $\overline{H}^*(\Gamma, L^2(Y; \mu))$.

5.2. Proof of theorem 1.5. We now prove theorem 1.5 which says that the real cohomology ring of a finitely generated nilpotent group is a quasi-isometry invariant. The proof follows from the following theorem, which is a "multiplicative" generalization of Shalom's [Sha, theorem 4.1.1] and its method of proof.

Theorem 5.1. *Let Γ be a finitely generated amenable group with the following property: Any unitary Γ -representation π with $\overline{H}^n(\Gamma, \pi) \neq 0$ contains the trivial representation. Let Λ be a finitely generated group which is quasi-isometric to Γ , and assume that the Betti numbers $b_n(\Lambda), b_n(\Gamma)$ are finite for all $n \geq 0$. Then there is a multiplicative injective homomorphism*

$$H^*(\Lambda, \mathbb{R}) \hookrightarrow H^*(\Gamma, \mathbb{R})$$

between the real cohomology rings of Λ and Γ .

First recall the well known fact that the classifying spaces of finitely generated torsionfree nilpotent groups are finite, and a finitely generated nilpotent group is virtually finitely generated torsionfree nilpotent, so the hypothesis on the Betti numbers is

satisfied. The fact that finitely generated nilpotent groups satisfy the representation-theoretic assumption of the theorem is proved in [Sha, theorem 4.1.3]. It is deduced from a theorem of Blanc which says that the continuous cohomology of a connected nilpotent Lie group with coefficients in an irreducible, non-trivial unitary representation always vanishes. So this is the only place where the nilpotence hypothesis of theorem 1.5 plays an essential role. Let Λ, Γ be finitely generated, nilpotent. Due to symmetry (and like in Shalom's article), theorem 5.1 yields that the Betti numbers of finitely generated nilpotent groups are the same. So the injective map $H^*(\Lambda; \mathbb{R}) \rightarrow H^*(\Gamma; \mathbb{R})$ in the preceding theorem must be an isomorphism. This completes the proof of 1.5 once we have shown the theorem above.

Proof of theorem 5.1. By theorem 2.2 there is a topological coupling X with compact fundamental domains X_Λ, X_Γ , and by theorem 2.4 we can equip X with a $\Gamma \times \Lambda$ -invariant non-trivial *ergodic* measure. By replacing Γ by a product $\Gamma \times F$ with a finite group (remark 2.3) we can assume that $X_\Gamma \subset X_\Lambda$. Note that this does not affect the assumption of the theorem. We write $\mathcal{F}(X_\Gamma)$ for $\mathcal{F}(X_\Gamma; \mathbb{R})$ and $L^2(X_\Gamma)$ for the real μ -square-integrable functions. In the sequel we use Shalom's explicit formula [Sha, section 3.2, (12)] for the induction rather than the abstract setup of section 3, since we work with the reduced cohomology. In terms of the standard homogeneous resolution Shalom's induction map $H^*(\Lambda, \mathbb{R}) \rightarrow H^*(\Gamma, L^2(X_\Lambda))$ is given by sending a cycle $w : \Lambda^{n+1} \rightarrow \mathbb{R}$ to $\omega' : \Gamma^{n+1} \rightarrow L^2(X_\Lambda)$ defined by

$$(5.4) \quad \omega'(\gamma_0, \dots, \gamma_n)(x) = \omega(\alpha(\gamma_0, x), \dots, \alpha(\gamma_n, x)).$$

Being continuous it descends to a map

$$I_{red}^* : \overline{H}^*(\Lambda, \mathbb{R}) \rightarrow \overline{H}(\Gamma, L^2(X_\Lambda)).$$

By [Sha, theorem 3.2.1] I_{red}^* is injective. Further note that $H^*(\Lambda, \mathbb{R}) = \overline{H}^*(\Lambda, \mathbb{R})$ and $H^*(\Lambda, \mathbb{R}) = \overline{H}^*(\Lambda, \mathbb{R})$ hold because, by hypothesis, the homology groups are finite-dimensional. Since $\{x; \alpha(\gamma, x) = \lambda\}$ is open and closed for fixed γ, λ , the functions $\omega'(\gamma_0, \dots, \gamma_n)$ actually lie in $\mathcal{F}(X_\Lambda)$ (compare with the proof of lemma 2.17). Hence I_{red}^* factorizes as

$$H^*(\Lambda, \mathbb{R}) \xrightarrow{I^*} H^*(\Gamma, \mathcal{F}(X_\Lambda)) \xrightarrow{j} \overline{H}^*(\Gamma, L^2(X_\Lambda)),$$

where j^* is the composition of the inclusion of coefficients and the canonical surjection $H^*(\Gamma, L^2(X_\Lambda)) \rightarrow \overline{H}^*(\Gamma, L^2(X_\Lambda))$. At this point we want to remark that it can be shown that this definition of I^* is the same as the one in section 3, but this not relevant here since we stick to Shalom's definition. Now let us collect all these data in the following

diagram:

$$\begin{array}{ccccc}
 & & \overline{H}^*(\Gamma, L^2(X_\Lambda)) & & \\
 & \nearrow I_{red}^* & \uparrow j^* & \searrow p_2^* & \\
 H^*(\Lambda, \mathbb{R}) & \xrightarrow{I^*} & H^*(\Gamma, \mathcal{F}(X_\Lambda)) & & \\
 & \searrow I^* \circ p_1^* & \downarrow i^* & \downarrow p_1^* & \\
 & & H^*(\Gamma, \mathbb{R}) & &
 \end{array}$$

The map i^* is the map induced by the inclusion of coefficients. The maps p_1^*, p_2^* are the maps induced by the (Γ -equivariant!) integration maps $L^2(X_\Lambda) \rightarrow \mathbb{R}, f \mapsto \int_{X_\Lambda} f d\mu$ resp. $\mathcal{F}(X_\Lambda) \rightarrow \mathbb{R}, f \mapsto \int_{X_\Lambda} f d\mu$.

Our aim is to show that the composition $p_1^* \circ I^*$ is multiplicative and injective. We have a direct sum $L^2(X_\Lambda) = \mathbb{R} \oplus L_0^2(X_\Lambda)$ of Γ -representations, where \mathbb{R} are the constant functions and $L_0^2(X_\Lambda)$ are the functions with mean value zero. Similarly, one obtains a direct sum decomposition $L^2(X_\Lambda; \mathbb{C}) = \mathbb{C} \oplus L_0^2(X_\Lambda; \mathbb{C})$ in the complex case and a $\mathbb{R}\Gamma$ -module decomposition $\mathcal{F}(X_\Lambda) = \mathbb{R} \oplus \mathcal{F}_0(X_\Lambda)$. Hence we get direct sum decompositions

$$\begin{aligned}
 \overline{H}^*(\Gamma, L^2(X_\Lambda)) &= \overline{H}^*(\Gamma, \mathbb{R}) \oplus \overline{H}^*(\Gamma, L_0^2(X_\Lambda)), \\
 H^*(\Gamma, \mathcal{F}(X_\Lambda)) &= H^*(\Gamma, \mathbb{R}) \oplus H^*(\Gamma, \mathcal{F}_0(X_\Lambda))
 \end{aligned}$$

which are respected by j^* . For an element $x \in H^m(\Gamma, \mathcal{F}(X_\Lambda))$ we use the notation $x = x_0 + x_1$, $x_0 \in H^m(\Gamma, \mathcal{F}_0(X_\Lambda))$, $x_1 \in H^m(\Gamma, \mathbb{R})$ for the sum decomposition. Since the Γ -action on X_Λ is ergodic, the Γ -representation $L_0^2(X_\Lambda; \mathbb{C})$ does not contain the trivial representation. So our assumption yields that $\overline{H}^n(\Gamma, L_0^2(X_\Lambda; \mathbb{C})) = 0$. Since $L_0^2(X_\Lambda; \mathbb{C}) = L_0^2(X_\Lambda) \oplus L_0^2(X_\Lambda)$ as real Γ -representations, it is $\overline{H}^n(\Gamma, L_0^2(X_\Lambda)) = 0$. In particular, p_2^* is an isomorphism. Since $p_1^* \circ I^* = p_2^* \circ I_{red}^*$ and I_{red}^* is injective, we obtain the injectivity of $p_1^* \circ I^*$.

From the formulas (5.2) and (5.4) one can see that I^* is multiplicative, so we are reduced to show that the map p_1^* is multiplicative in our special situation (in general, it is not since it is induced by integration!). The map j is obviously a module homomorphism with respect to both the left and right $H^*(\Gamma, \mathcal{F}(X_\Lambda))$ -module structure on $H^*(\Gamma, \mathcal{F}(X_\Lambda))$ resp. on $\overline{H}^*(\Gamma, L^2(X_\Lambda))$. For the module structure see the preceding section. Since p_2^* is an isomorphism, it follows that $\ker p_1^* = \ker j^*$, and hence $\ker p_1^*$ is a two-sided ideal of the cohomology ring $H^*(\Gamma, \mathcal{F}(X_\Lambda))$. Further note $\ker p_1^* = H^*(\Gamma, \mathcal{F}_0(X_\Lambda))$. Now let $x \in H^m(\Gamma, \mathcal{F}(X_\Lambda))$, $x' \in H^n(\Gamma, \mathcal{F}(X_\Lambda))$, and consider the sum decompositions $x = x_0 + x_1$, $x' = x'_0 + x'_1$ as above. It is clear that $p_1^{m+n}(x_1 \cup x'_1) = p_1^m(x_1) \cup p_1^n(x'_1)$ since i^* , induced by the ring homomorphism $\mathbb{R} \rightarrow \mathcal{F}(X_\Lambda)$, is a multiplicative map. Now one

computes:

$$\begin{aligned}
p_1^{m+n}(x \cup x') &= p_1^{m+n}((x_0 + x_1) \cup (x'_0 + x'_1)) \\
&= p_1^{m+n}(x_0 \cup x'_0) + p_1^{m+n}(x_0 \cup x'_1) + p_1^{m+n}(x_1 \cup x'_0) + p_1^{m+n}(x_1 \cup x'_1) \\
&= p_1^{m+n}(x_1 \cup x'_1) \\
&= p_1^m(x_1) \cup p_1^n(x'_1) \\
&= p_1^m(x) \cup p_1^n(x').
\end{aligned}$$

□

5.3. Remarks. The \mathbb{Q} -Malcev completions of quasi-isometric nilpotent groups are in general not isomorphic. In fact, they are only isomorphic if the groups are commensurable. This fact is also reflected in cohomology: We give examples of quasi-isometric nilpotent groups whose rational cohomology rings are not isomorphic as \mathbb{Q} -algebras. So theorem 1.5 does not hold for rational coefficients. Consider the $2n + 1$ -dimensional Heisenberg Lie algebra $H(n, K)$ over a field K . This is the Lie algebra of matrices with entries in K of the form

$$\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

where x, y are an n -dimensional row resp. column vector and $z \in K$. Now consider the $4n + 2$ -dimensional, *rational* Lie algebras $\mathfrak{g} = H(n, \mathbb{Q}(\sqrt{2}))$ and $\mathfrak{h} = H(n, \mathbb{Q}) \oplus H(n, \mathbb{Q})$. Over \mathbb{Q} , \mathfrak{g} and \mathfrak{h} are not isomorphic since the \mathbb{Q} -codimension of the centralizers of elements in \mathfrak{g} is either 0 or ≥ 2 , whereas in \mathfrak{h} there is an element whose centralizer has \mathbb{Q} -codimension 1. However, $\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{R}$ are easily seen to be isomorphic as real Lie algebras. This example is discussed in [Rag72, remark 2.15]. Hence $\mathfrak{g}, \mathfrak{h}$ are two different rational structures of the same real Lie algebra, thereby defining two quasi-isometric (but not commensurable) nilpotent Lie groups Λ and Γ . By virtue of Nomizu's theorem [Nom54] $H^*(\Lambda, \mathbb{Q})$ and $H^*(\mathfrak{g}, \mathbb{Q})$ resp. $H^*(\Gamma, \mathbb{Q})$ and $H^*(\mathfrak{h}, \mathbb{Q})$ are isomorphic as algebras. Now assume that $n \geq 2$. Due to [CT95, example 2.3] the Chevalley-Eilenberg complexes of $H(n, K)$, hence also of $H(n, K) \oplus H(n, K)$, as differential graded algebras are 1-formal, in the sense of rational homotopy theory. Note that 1-formality is independent of the ground field. The reason behind this is that these algebras are quadratically presented. In particular, the rational Lie algebras $\mathfrak{g}, \mathfrak{h}$ can be reconstructed from their rational cohomology algebras (compare with [CT95, prop. 2.1]). Therefore the latter cannot be isomorphic. At this point we remark that the class of quadratically presented Lie algebras is quite restricted within the class of all nilpotent Lie algebras. See [CT95] for more information.

Next we discuss what information we can get if the representation theoretic assumption of theorem 5.1 is not satisfied in every dimension but only for the first reduced cohomology. Explicitly, we assume only that any unitary Γ -representation π with $\overline{H}^1(\Gamma, \pi) \neq 0$ contains the trivial representation. This property of Γ was called *property H_T* in [Sha, definition of p. 7]. Otherwise we retain the same assumptions on Λ and Γ as in 5.1. By

the same arguments we get homomorphisms $\alpha^n : H^n(\Lambda, \mathbb{R}) \rightarrow H^n(\Gamma, \mathbb{R})$, where α^1 is injective. Further, if $x \in H^1(\Lambda, \mathbb{R})$, $y \in H^n(\Lambda, \mathbb{R})$, $n \geq 0$, then

$$\alpha^{n+1}(x \cup y) = \alpha^1(x) \cup \alpha^n(y)$$

holds true. To show that, we must verify that the integration map p_1^* in the proof satisfies

$$(5.5) \quad p_1^{n+1}(z \cup w) = p_1^1(z) \cup p_1^n(w)$$

for $z \in H^1(\Gamma, \mathcal{F}(X_\Lambda))$, $w \in H^n(\Gamma, \mathcal{F}(X_\Lambda))$. Decompose $z = z_0 + z_1$ as in the proof. Then one can use the module structure on the reduced cohomology to conclude $p_1^{n+1}(z_0 \cup w) = 0$. On the other hand, $p_1^{n+1}(z_1 \cup w) = p_1^1(z_1) \cup p_1^n(w)$ since p_1 is obviously a $H^*(\Gamma, \mathbb{R})$ -module homomorphism. Hence (5.5) follows.

6. QUASI-ISOMETRY INVARIANCE OF NOVIKOV-SHUBIN INVARIANTS

6.1. Review of the definition of Novikov-Shubin invariants. In this section we review the usual definition of Novikov-Shubin invariants and then the *homological viewpoint* developed in [LRS99]. The latter is strongly inspired by Lück's algebraic approach to L^2 -Betti numbers.

Let Γ be a discrete group. The Hilbert space with basis Γ is denoted by $l^2(\Gamma)$. The bounded operators on $l^2(\Gamma)$ which are equivariant with respect to the left Γ -action on $l^2(\Gamma)$ form a von Neumann algebra $\mathcal{N}(\Gamma)$, called the **group von Neumann algebra**. Equivalently, $\mathcal{N}(\Gamma)$ can be defined as the weak closure of $\mathbb{C}\Gamma$ whose elements act on $l^2(\Gamma)$ by right multiplication. A von Neumann algebra \mathcal{A} is called **finite** if \mathcal{A} comes equipped with a **finite trace**, i.e. with a finite, normal, faithful trace $\text{tr}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathbb{C}$. The examples of finite von Neumann algebras we will consider are the following.

- The group von Neumann algebra $\mathcal{N}(\Gamma)$ with its standard trace $\text{tr}_{\mathcal{N}(\Gamma)} : \mathcal{N}(\Gamma) \rightarrow \mathbb{C}$ given by $\text{tr}_{\mathcal{N}(\Gamma)}(T) = \langle 1_\Gamma, T(1_\Gamma) \rangle_{l^2(\Gamma)}$. In particular, the trace of an element in $\mathbb{C}\Gamma$ is the coefficient of the unit element.
- Let X be a standard Borel space equipped with a probability Borel measure μ . Then $L^\infty(X) = L^\infty(X; \mu)$ is a finite von Neumann algebra with the trace $\text{tr}_{L^\infty(X)}(f) = \int_X f d\mu$.
- Assume X is additionally equipped with a μ -preserving Γ -action. Then there is the **von Neumann crossed product** $L^\infty(X) \bar{\rtimes} \Gamma$ which contains the *algebraic* crossed product $L^\infty(X) \rtimes \Gamma$ as a weakly dense subalgebra. Further, $L^\infty(X) \bar{\rtimes} \Gamma$ has a finite trace whose restriction to $L^\infty(X) \rtimes \Gamma$ is given by $\text{tr}(\sum_\gamma f_\gamma \gamma) = \int_X f_1 d\mu$.
- If \mathcal{A} is a finite von Neumann algebra with trace $\text{tr}_{\mathcal{A}}$, then the n -dimensional square matrices $M_n(\mathcal{A})$ over \mathcal{A} form again a finite von Neumann algebra with the trace

$$\text{tr}_{M_n(\mathcal{A})}((T_{ij})_{1 \leq i, j \leq n}) = \frac{1}{n} \sum_{i=1}^n \text{tr}_{\mathcal{A}}(T_{ii}).$$

Any finite von Neumann algebra \mathcal{A} is contained in an algebra of unbounded operators which we explain next. A not necessarily bounded operator a on $\mathcal{B}(H)$ is **affiliated** to $\mathcal{A} \subset \mathcal{B}(H)$ if $ba \subset ab$ for all operators $b \in \mathcal{A}'$ holds. Here \mathcal{A}' is the commutant

of \mathcal{A} , and $ba \subset ab$ means that restricted to the possibly smaller domain of ba the two operators coincide.

Definition 6.1. Let $\mathcal{U}(\mathcal{A})$ denote the set of all closed densely defined operators affiliated to \mathcal{A} . If \mathcal{A} happens to be $\mathcal{N}(\Gamma)$ we write $\mathcal{U}(\Gamma)$ instead of $\mathcal{U}(\mathcal{N}(\Gamma))$.

Murray and von Neumann [Mv36] showed that these unbounded operators form a $*$ -algebra containing \mathcal{A} , when addition, multiplication and involution are defined as the closures of the naive addition, multiplication and involution. More information about $\mathcal{U}(\mathcal{A})$ can be found in [Lüc02, chapter 8] and [Rei01]. From the latter reference we take also the following result.

Theorem 6.2. *The inclusion $\mathcal{A} \subset \mathcal{U}(\mathcal{A})$ is a flat ring extension.*

Definition 6.3. Let \mathcal{A} be a finite von Neumann algebra with trace $\text{tr}_{\mathcal{A}}$, and for a normal operator $T \in \mathcal{A}$ we denote by $f(T)$ the operator obtained from spectral calculus with respect to the function f . The **Novikov-Shubin invariant** $\alpha_{\mathcal{A}}(T) \in [0, \infty] \cup \infty^+$ of a **positive operator** $T \in \mathcal{A}$ is defined as

$$\alpha_{\mathcal{A}}(T) = \begin{cases} \liminf_{\lambda \rightarrow 0^+} \frac{\ln(\text{tr}_{\mathcal{A}}(\chi_{(0, \lambda]}(T)))}{\ln(\lambda)} \in [0, \infty] & \text{if } \text{tr}_{\mathcal{A}}(\chi_{(0, \lambda]}(T)) > 0 \text{ for } \lambda > 0, \\ \infty^+ & \text{otherwise.} \end{cases}$$

Here $\chi_{(0, \lambda]}$ is the characteristic function of the interval $(0, \lambda]$.

Now we explain the usual definition of the Novikov-Shubin invariant of a discrete group Γ having a finite type model of its classifying space $E\Gamma$. The chain complex of such a model is a free $\mathbb{Z}\Gamma$ -resolution of \mathbb{Z}

$$0 \leftarrow \mathbb{Z} \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \xleftarrow{d_3} \dots,$$

where for each $i \geq 0$ the module F_i is a finitely generated free $\mathbb{Z}\Gamma$ -module, let us say of rank $r(i)$. Now by tensoring this complex with $l^2(\Gamma)$ we get a Hilbert complex $l^2(\Gamma) \otimes_{\mathbb{Z}\Gamma} F_*$ whose operators are bounded Γ -equivariant operators \bar{d}_* . Then, by choosing a basis for each F_i , we can consider $\bar{d}_i^* \bar{d}_i$ as an element in $M_{r(i)}(\mathcal{N}(\Gamma))$. Now the following makes sense and is indeed well defined (see [Lüc02, chapter 2] for proofs and much more information).

Definition 6.4 (classical definition). The i -th Novikov-Shubin invariant $\alpha_i(\Gamma)$, $i \geq 1$, of Γ is defined as $\alpha_{M_{r(i)}(\mathcal{N}(\Gamma))}(\bar{d}_i^* \bar{d}_i)$.

For the homological viewpoint one defines the notion of capacity for any $\mathcal{N}(\Gamma)$ -module in the first place, and then the Novikov-Shubin invariants of Γ are essentially the reciprocals of the capacities of the group homology of Γ with coefficients in the $\mathbb{C}\Gamma$ -module $\mathcal{N}(\Gamma)$. To a module M over a finite von Neumann algebra \mathcal{A} there are two numerical invariants attached, the **dimension** $\dim_{\mathcal{A}}(M) \in [0, \infty]$ in the sense of Lück [Lüc98a, Lüc98b] and the **capacity** $c_{\mathcal{A}}(M) \in [0, \infty] \cup \{0^-\}$ [LRS99]. We will not elaborate upon neither the dimension nor the question of $c_{\mathcal{A}}(M)$ being well defined; the reader is referred to consult [Lüc02] for the first and [LRS99] for the latter. The definition of $c_{\mathcal{A}}(M)$ proceeds in three steps.

1. STEP: Let M be a finitely presented \mathcal{A} -module M with $\dim_{\mathcal{A}}(M) = 0$. By [Lüc97, lemma 3.4, definition 3.11] there is a short exact sequence

$$\mathcal{A}^n \xrightarrow{r_A} \mathcal{A}^n \twoheadrightarrow M,$$

where r_A is right multiplication with a positive matrix $A \in M_n(\mathcal{A})$. Define

$$c_{\mathcal{A}}(M) = \frac{1}{\alpha_{M_n(\mathcal{A})}(A)} \in [0, \infty] \cup \{0^-\},$$

where we agree on the following rules

$$1/\infty = 0, 1/\infty^+ = 0^- \text{ and } 0^- < 0.$$

2. STEP: An \mathcal{A} -module M is called **measurable** if it is the quotient of a finitely presented \mathcal{A} -module N with $\dim_{\mathcal{A}}(N) = 0$. For such a module M we define $c'_{\mathcal{A}}(M)$ as $c'_{\mathcal{A}}(M) = \inf\{c_{\mathcal{A}}(N); N \text{ finitely presented, } \dim_{\mathcal{A}}(N) = 0, N \text{ surjects onto } M\}$.
3. STEP: For an arbitrary \mathcal{A} -module M we set

$$c''_{\mathcal{A}}(M) = \sup\{c_{\mathcal{A}}(N); N \text{ measurable submodule of } M\}.$$

In [LRS99, proposition 2.4] it is shown that for a finitely presented \mathcal{A} -module M with $\dim_{\mathcal{A}}(M) = 0$ and a measurable \mathcal{A} -module N one has $c_{\mathcal{A}}(M) = c'_{\mathcal{A}}(M)$ and $c'_{\mathcal{A}}(N) = c''_{\mathcal{A}}(N)$. Therefore we do not need to distinguish between $c_{\mathcal{A}}, c'_{\mathcal{A}}, c''_{\mathcal{A}}$. We call $c_{\mathcal{A}}(M)$ the **capacity** of M .

Definition 6.5. An \mathcal{A} -module M is **cofinal-measurable** if each finitely generated submodule is measurable. This is equivalent to $\mathcal{U}(\mathcal{A}) \otimes_{\mathcal{A}} M = 0$ [Rei01, note 4.5].

Definition 6.6 (homological approach). Let Γ be an arbitrary discrete group. Then we define the i -th **capacity** $c_i(\Gamma)$ of Γ as

$$c_i(\Gamma) = c_{\mathcal{N}(\Gamma)}(H_i(\Gamma, \mathcal{N}(\Gamma))), i \geq 0.$$

The i -th **Novikov-Shubin invariant** $\alpha_i(\Gamma)$, $i \geq 1$, of Γ is defined as the inverse of $c_{i-1}(\Gamma)$ with the usual rules $1/0 = \infty$, $1/0^- = \infty^+$.

Of course, this can be shown to be compatible with the classical definition whenever it applies. What is the point about introducing a name for the reciprocal of the Novikov-Shubin invariant? In some sense, $c_i(\Gamma)$ measures the *size* of $H_i(\Gamma, \mathcal{N}(\Gamma))$ if the i -th L^2 -Betti number vanishes. In particular, it equals 0^- if $H_i(\Gamma, \mathcal{N}(\Gamma)) = 0$. For this reason, it is more convenient in the homological setting to work with the inverse. As already indicated in the introduction, the definition of the capacity is not well-behaved for arbitrary Γ . When we restrict to amenable groups, here is the subclass (see theorem 6.12 for its description) where $c_i(\Gamma)$ still behaves well:

Definition 6.7. Let \mathcal{CM} be the class of all finitely generated amenable groups Γ such that $H_n(\Gamma, \mathcal{N}(\Gamma))$ is cofinal-measurable for $n \geq 1$. Equivalently, we can say that \mathcal{CM} consists precisely of all finitely generated amenable groups Γ with $H_n(\Gamma, \mathcal{U}(\Gamma)) = 0$ for $n \geq 1$.

6.2. Some technical properties of the capacity.

Theorem 6.8. *Let $\mathcal{A} \subset \mathcal{B}$ be a trace-preserving inclusion of finite von Neumann algebras, and let M be an \mathcal{A} -module. Then the following holds.*

- (i) *The ring extensions $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{U}(\mathcal{A}) \subset \mathcal{U}(\mathcal{B})$ are faithfully flat.*
- (ii) *If M is a cofinal-measurable \mathcal{A} -module, then $\mathcal{B} \otimes_{\mathcal{A}} M$ is cofinal-measurable and $c_{\mathcal{A}}(M) = c_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} M)$ holds.*
- (iii) *If M is a finitely presented \mathcal{A} -module, then $\mathcal{B} \otimes_{\mathcal{A}} M$ is finitely presented and $c_{\mathcal{A}}(M) = c_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} M)$ holds.*

Proof.

(i) In [Lüc02, 6.29] the first statement in (i) is proved for inclusions of group von Neumann algebras induced by inclusions of groups. We give a (shorter) proof of the general case. We show that \mathcal{B} is a torsionless \mathcal{A} -module, i.e. for every $b \in \mathcal{B}$, $b \neq 0$, there exists an \mathcal{A} -linear map $f : \mathcal{B} \rightarrow \mathcal{A}$ with $f(b) \neq 0$. Any von Neumann algebra is a semihereditary ring [Lüc02, remark on p. 288], and every torsionless module over a semihereditary ring is flat [Lam99, theorem 4.67], hence this implies that $\mathcal{A} \subset \mathcal{B}$ is flat. Let b be a non-zero element in \mathcal{B} . Since b is the sum of four unitaries [KR97, theorem 4.1.7 on p. 242], there is a unitary $u \in \mathcal{B}$ such that $\text{tr}_{\mathcal{B}}(b^*u) \neq 0$. The map $\mathcal{A} \rightarrow \mathcal{B}$, $a \mapsto au$ extends to an \mathcal{A} -equivariant isometric embedding $i : l^2\mathcal{A} \rightarrow l^2\mathcal{B}$ between the GNS-representations of $\text{tr}_{\mathcal{A}}$ resp. $\text{tr}_{\mathcal{B}}$. Taking the orthogonal projection onto the image of i yields an \mathcal{A} -equivariant bounded split $f : l^2\mathcal{B} \rightarrow l^2\mathcal{A}$ of i . Since $\langle b, u \rangle_{l^2\mathcal{B}} = \text{tr}(b^*u) \neq 0$ we get $f(b) \neq 0$. It remains to show that $f(\mathcal{B}) \subset \mathcal{A}$. Due to a theorem of Dixmier [Lüc02, theorem 9.9], this follows from the fact that $\mathcal{A} \rightarrow l^2\mathcal{A}$, $n \mapsto n \cdot f(m) = f(n \cdot m)$ extends to a bounded operator on $l^2\mathcal{A}$. The construction of f for $u = 1$ defines an \mathcal{A} -equivariant split of the inclusion $\mathcal{A} \subset \mathcal{B}$. Hence $\mathcal{A} \subset \mathcal{B}$ is a faithfully flat ring extension. The algebra of affiliated operators $\mathcal{U}(\mathcal{A})$ is a von Neumann regular ring [Rei01, 2.1] (see also [Lüc02, theorem 8.22]). Hence every $\mathcal{U}(\mathcal{A})$ -module is flat [Lam99, theorem 4.21]. This implies also that every ring extension $\mathcal{U}(\mathcal{A}) \subset R$ into a bigger ring R is faithfully flat [Lam99, theorem 4.74 (4)].

(ii) This is proved in [LRS99, lemma 2.12] for inclusions of group von Neumann algebras but the proof carries over to arbitrary inclusions without modifications.

(iii) From the exactness of $\mathcal{A} \subset \mathcal{B}$ follows immediately that $\mathcal{B} \otimes_{\mathcal{A}} M$ is again finitely presented. We use the fact that a finitely presented \mathcal{A} -module M splits as $M = \mathbf{P}M \oplus \mathbf{T}M$ into a finitely generated projective module $\mathbf{P}M$ and into a finitely presented module $\mathbf{T}M$ with $\dim_{\mathcal{A}}(\mathbf{T}M) = 0$ [Lüc98a, theorem 0.6]. The capacity satisfies $c_{\mathcal{A}}(M) = c_{\mathcal{A}}(\mathbf{T}M)$ by [LRS99, theorem 2.7]. Similarly, we obtain $c_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} M) = c_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathbf{T}M)$. Since $\mathbf{T}M$ is cofinal-measurable, (iii) follows now from (ii). \square

Theorem 6.9. *Let \mathcal{A} be a finite von Neumann algebra and $p \in \mathcal{A}$ a full projection. Then for every $p\mathcal{A}p$ -module M the equality $c_{\mathcal{A}}(\mathcal{A}p \otimes_{p\mathcal{A}p} M) = c_{p\mathcal{A}p}(M)$ holds.*

Proof. First assume that M is zero-dimensional and finitely presented. Then there exists a short exact sequence [Lüc97, lemma 3.4] of the form

$$(p\mathcal{A}p)^n \xrightarrow{r_A} (p\mathcal{A}p)^n \twoheadrightarrow M.$$

where r_A is right multiplication with a positive matrix $A \in M_n(pAp)$. Tensoring by Ap is flat (see remark 2.20), so we obtain a short exact sequence

$$(Ap)^n \oplus (\mathcal{A}(1-p))^n \xrightarrow{c_{r_A \oplus id}} (Ap)^n \oplus (\mathcal{A}(1-p))^n \twoheadrightarrow Ap \otimes_{pAp} M.$$

The map in the middle is an \mathcal{A} -equivariant endomorphism of \mathcal{A}^n given by right multiplication with the matrix $A + (1-p) \text{id}_{\mathcal{A}^n} \in M_n(\mathcal{A})$. In particular, we have that $Ap \otimes_{pAp} M$ is again zero-dimensional since the dimension $\dim_{\mathcal{A}}$ is additive [Lüc98a, theorem 0.6]. Now let us show two general facts about spectral calculus in a von Neumann algebra \mathcal{A} . Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an essentially bounded Borel function with $f(0) = 0$. (1) For selfadjoint $a, b \in \mathcal{A}$ satisfying $a \cdot b = b \cdot a = 0$ we have the following equality of operators obtained by spectral calculus in \mathcal{A} with respect to f .

$$f(a+b; \mathcal{A}) = f(a; \mathcal{A}) + f(b; \mathcal{A})$$

(2) For a projection $p \in \mathcal{A}$ and selfadjoint $a \in pAp$ the equality

$$f(a; \mathcal{A}) = pf(a; \mathcal{A})p = f(a; pAp)$$

holds. Since $f(a; \mathcal{A})$ is the strong limit of operators of the form $q(a)$ where q is a polynomial with vanishing constant term, it suffices to show these equalities for $f(x) = x^n$, $n \geq 1$. This is obvious; note for (2) that $pa = ap = a$ holds. Notice that we have $\chi_{(0, \lambda]}((1-p) \text{id}_{\mathcal{A}^n}) = 0$ for $\lambda < 1$, because $(1-p) \text{id}_{\mathcal{A}^n}$ is a projection. Now we compute using (1) and (2) for $\lambda < 1$

$$\begin{aligned} \text{tr}_{M_n(\mathcal{A})}(\chi_{(0, \lambda]}(A + (1-p) \text{id}_{\mathcal{A}^n}; \mathcal{A})) &= \text{tr}_{M_n(\mathcal{A})}(\chi_{(0, \lambda]}(A; \mathcal{A})) \\ &= \text{tr}_{M_n(\mathcal{A})}(\chi_{(0, \lambda]}(A; pAp)) \\ &= \text{tr}_{\mathcal{A}}(p) \cdot \text{tr}_{M_n(pAp)}(\chi_{(0, \lambda]}(A; pAp)). \end{aligned}$$

Thus $\alpha_{M_n(pAp)}(A) = \alpha_{M_n(\mathcal{A})}(A + (1-p) \text{id}_{\mathcal{A}^n})$ holds, hence the assertion for M follows.

Now assume that M is measurable. For $\epsilon > 0$ let N be a zero-dimensional, finitely presented pAp -module surjecting onto M such that $c_{pAp}(N) \leq c_{pAp}(M) - \epsilon$. As seen above, $Ap \otimes_{pAp} N$ is again finitely presented zero-dimensional, and it surjects onto $Ap \otimes_{pAp} M$. Then we obtain

$$c_{\mathcal{A}}(Ap \otimes_{pAp} M) \leq c_{\mathcal{A}}(Ap \otimes_{pAp} N) = c_{pAp}(N) \leq c_{pAp}(M) - \epsilon,$$

thus $c_{\mathcal{A}}(Ap \otimes_{pAp} M) \leq c_{pAp}(M)$. Since the Morita equivalence $Ap \otimes_{pAp} -$ has the inverse $p\mathcal{A} \otimes_{\mathcal{A}} -$ we can conclude analogously to obtain $c_{pAp}(M) \leq c_{\mathcal{A}}(Ap \otimes_{pAp} M)$, thus $c_{pAp}(M) = c_{\mathcal{A}}(Ap \otimes_{pAp} M)$. The argument for an arbitrary module M is similar. \square

Corollary 6.10. *Let \mathcal{A} be a finite von Neumann algebra and $p \in \mathcal{A}$ a full projection. Then for every \mathcal{A} -module M the equality $c_{\mathcal{A}}(M) = c_{pAp}(pM)$ holds.*

6.3. The proof of theorem 1.6 and the class \mathcal{CM} .

Proof of theorem 1.6. Let Γ be a group in \mathcal{CM} and Λ be quasi-isometric to Γ . According to 6.12 (ii) Λ lies then also in \mathcal{CM} . In particular both groups are amenable. According to theorems 2.2 and 2.4 there exists a topological coupling X of Γ, Λ with compact fundamental domains X_{Γ}, X_{Λ} and a non-trivial $\Gamma \times \Lambda$ -invariant Borel measure μ on X

such that X_Γ, X_Λ have finite measure. Due to lemma 2.17 there exist open and closed subspaces $A \subset X_\Gamma, B \subset X_\Lambda$ for which there is a trace-preserving ring isomorphism between $\chi_A \mathcal{F}(X_\Gamma; \mathbb{C}) \rtimes \Lambda_{\chi_A}$ and $\chi_B \mathcal{F}(X_\Lambda; \mathbb{C}) \rtimes \Gamma_{\chi_B}$. Further, χ_A resp. χ_B is a full idempotent in its respective crossed product ring and hence in any bigger ring according to lemma 2.21. From the construction of X_Γ as a closed-open subset of some space of functions between discrete sets it is clear that the closed-open sets of X_Γ form a basis of the topology. Hence $\mathcal{F}(X_\Gamma; \mathbb{C}) \subset L^\infty(X_\Gamma)$ is weakly dense. The same applies to X_Λ . Since the isomorphism above is trace-preserving, it extends to an isomorphism

$$\chi_A L^\infty(X_\Gamma) \rtimes \Lambda_{\chi_A} \rightarrow \chi_B L^\infty(X_\Lambda) \rtimes \Gamma_{\chi_B}.$$

For the same reason this extends to a trace-preserving isomorphism of the respective von Neumann algebras and to an isomorphism of their algebras of affiliated operators. So we come up with the following commutative diagram with vertical isomorphisms.

(6.1)

$$\begin{array}{ccccccc} L^\infty(A) & \hookrightarrow & \chi_A L^\infty(X_\Gamma) \rtimes \Lambda_{\chi_A} & \hookrightarrow & \chi_A L^\infty(X_\Gamma) \bar{\rtimes} \Lambda_{\chi_A} & \hookrightarrow & \mathcal{U}(\chi_A L^\infty(X_\Gamma) \bar{\rtimes} \Lambda_{\chi_A}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ L^\infty(B) & \hookrightarrow & \chi_B L^\infty(X_\Lambda) \rtimes \Gamma_{\chi_B} & \hookrightarrow & \chi_B L^\infty(X_\Lambda) \bar{\rtimes} \Gamma_{\chi_B} & \hookrightarrow & \mathcal{U}(\chi_B L^\infty(X_\Lambda) \bar{\rtimes} \Gamma_{\chi_B}) \end{array}$$

Hence it suffices to prove that

$$(6.2) \quad \begin{aligned} c_n(\Lambda) &= c_{\chi_A L^\infty(X_\Gamma) \bar{\rtimes} \Lambda_{\chi_A}} \left(\text{Tor}_n^{\chi_A L^\infty(X_\Gamma) \rtimes \Lambda_{\chi_A}} (\chi_A L^\infty(X_\Gamma) \bar{\rtimes} \Lambda_{\chi_A}, L^\infty(A)) \right) \\ c_n(\Gamma) &= c_{\chi_B L^\infty(X_\Lambda) \bar{\rtimes} \Gamma_{\chi_B}} \left(\text{Tor}_n^{\chi_B L^\infty(X_\Lambda) \rtimes \Gamma_{\chi_B}} (\chi_B L^\infty(X_\Lambda) \bar{\rtimes} \Gamma_{\chi_B}, L^\infty(B)) \right), \end{aligned}$$

which follows by applying (in this order) theorem 6.8 (ii), theorem 6.8 (i), (A.4) and (A.6) from the appendix and corollary 6.10:

$$\begin{aligned} c_n(\Lambda) &= c_{\mathcal{N}(\Lambda)} \left(H_n(\Gamma, \mathcal{N}(\Lambda)) \right) \\ &= c_{\mathcal{N}(\Lambda)} \left(\text{Tor}_n^{\mathbb{C}\Lambda}(\mathcal{N}(\Lambda), \mathbb{C}) \right) \\ &= c_{L^\infty(X_\Gamma) \bar{\rtimes} \Lambda} \left(L^\infty(X_\Gamma) \bar{\rtimes} \Lambda \otimes_{\mathcal{N}(\Lambda)} \text{Tor}_n^{\mathbb{C}\Lambda}(\mathcal{N}(\Lambda), \mathbb{C}) \right) \\ &= c_{L^\infty(X_\Gamma) \bar{\rtimes} \Lambda} \left(\text{Tor}_n^{\mathbb{C}\Lambda}(L^\infty(X_\Gamma) \bar{\rtimes} \Lambda, \mathbb{C}) \right) \\ &= c_{L^\infty(X_\Gamma) \bar{\rtimes} \Lambda} \left(\text{Tor}_n^{L^\infty(X_\Gamma) \rtimes \Lambda}(L^\infty(X_\Gamma) \bar{\rtimes} \Lambda, L^\infty(X_\Lambda)) \right) \\ &= c_{L^\infty(X_\Gamma) \bar{\rtimes} \Lambda} \left(\text{Tor}_n^{\chi_A L^\infty(X_\Gamma) \rtimes \Lambda_{\chi_A}}(L^\infty(X_\Gamma) \bar{\rtimes} \Lambda_{\chi_A}, L^\infty(A)) \right) \\ &= c_{\chi_A L^\infty(X_\Gamma) \bar{\rtimes} \Lambda_{\chi_A}} \left(\chi_A \text{Tor}_n^{\chi_A L^\infty(X_\Gamma) \rtimes \Lambda_{\chi_A}}(L^\infty(X_\Gamma) \bar{\rtimes} \Lambda_{\chi_A}, L^\infty(A)) \right) \\ &= c_{\chi_A L^\infty(X_\Gamma) \bar{\rtimes} \Lambda_{\chi_A}} \left(\text{Tor}_n^{\chi_A L^\infty(X_\Gamma) \rtimes \Lambda_{\chi_A}}(\chi_A L^\infty(X_\Gamma) \bar{\rtimes} \Lambda_{\chi_A}, L^\infty(A)) \right). \end{aligned}$$

Similar for Γ . This finishes the proof of theorem 1.6. □

Basically the same method yields also the following result for not necessarily amenable groups, which should be compared to the proportionality theorem in [Lüc02, theorem 3.183].

Theorem 6.11. *Let Γ and Λ be finitely generated groups, and assume that Γ is of type FP_{n+1} over \mathbb{C} . If Γ and Λ are cocompact lattices of the same locally compact group, then $\alpha_i(\Gamma) = \alpha_i(\Lambda)$ holds for $1 \leq i \leq n + 1$.*

We indicate how to modify the proof above. Details are left to the reader. By [Alo94, corollary 9] Λ is also of type FP_{n+1} over \mathbb{C} . This assumption is needed for the compatibility of the capacity with induction from the group von Neumann algebra to a bigger one (see theorem 6.8 (iii)) and replaces the hypothesis “being in \mathcal{C} ” needed to apply 6.8 (ii). The topological coupling used for the proof is the common locally compact group that contains Λ, Γ as cocompact lattices.

Theorem 6.12.

- (i) *The class \mathcal{CM} contains all finitely generated elementary amenable groups that have a bound on the orders of their finite subgroups and all amenable groups of type FP_∞ over \mathbb{C} . Moreover, if a finitely generated amenable group Γ has an infinite, finitely generated, normal subgroup in \mathcal{CM} , then Γ lies in \mathcal{CM} .*
- (ii) *The class \mathcal{CM} is geometric, i.e. if a group Γ is quasi-isometric to a group in \mathcal{CM} then Γ lies in \mathcal{CM} .*

Proof. (i) Let Γ be an amenable group of type FP_∞ over \mathbb{C} . Since the category of finitely presented modules over a finite von Neumann algebra is abelian [Lüc97, theorem 0.2], it follows that $H_n(\Gamma; \mathcal{N}(\Gamma))$ is a finitely presented $\mathcal{N}(\Gamma)$ -module. By a theorem of Cheeger and Gromov $\dim_{\mathcal{N}(\Gamma)}(H_n(\Gamma; \mathcal{N}(\Gamma))) = 0$, $n \geq 1$, holds for amenable Γ (see also [Lüc98a, corollary 5.13]). By [Lüc02, 8.22 (4)], $\dim_{\mathcal{N}(\Gamma)}(M) = 0$ and $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} M = 0$ are equivalent for a finitely presented module M . This yields $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_n(\Gamma; \mathcal{N}(\Gamma)) = 0$, $n \geq 1$. By theorem 6.2 we obtain $H_n(\Gamma; \mathcal{U}(\Gamma)) = 0$, $n \geq 1$, hence Γ lies in the class \mathcal{CM} . For an elementary amenable group with a bound on the orders of finite subgroups $\mathcal{U}(\Gamma)$ is a flat $\mathbb{C}G$ -module [Rei99, theorem 9.1]. Hence such a Γ lies in \mathcal{CM} . Now assume that a group Γ has an infinite, finitely generated, normal subgroup Λ which lies in \mathcal{CM} , i.e. $H_n(\Lambda; \mathcal{U}(\Lambda)) = 0$, $n \geq 1$. Since Λ is finitely generated, the module $H_0(\Lambda; \mathcal{N}(\Lambda))$ is finitely presented, and since Λ is infinite, its zeroth L^2 -Betti number vanishes. As seen above, this implies $H_0(\Lambda; \mathcal{U}(\Lambda)) = 0$. Since $\mathcal{U}(\Gamma)$ is flat over $\mathcal{U}(\Lambda)$ by 6.8 we get $H_n(\Lambda; \mathcal{U}(\Gamma)) = 0$ for $n \geq 0$. The Hochschild-Serre spectral sequence now yields $H_n(\Gamma; \mathcal{U}(\Gamma)) = 0$, $n \geq 0$, in particular $\Gamma \in \mathcal{CM}$.

(ii) Let be $\Gamma \in \mathcal{CM}$, and Λ be quasi-isometric to Γ . The argument is completely analogous to the one for equation (6.2): It turns out that the vanishing of $H_n(\Gamma, \mathcal{U}(\Gamma))$ is equivalent to the vanishing of

$$\mathrm{Tor}_n^{\chi_B L^\infty(X_\Lambda) \rtimes \Gamma \chi_B} \left(\underbrace{\mathcal{U}(\chi_B L^\infty(X_\Lambda) \rtimes \Gamma \chi_B)}_{=\chi_B \mathcal{U}(L^\infty(X_\Lambda) \rtimes \Gamma) \chi_B}, L^\infty(B) \right),$$

and similar for Λ . Due to diagram (6.1), $H_n(\Gamma, \mathcal{U}(\Gamma))$ vanishes if and only if $H_n(\Lambda, \mathcal{U}(\Lambda))$ vanishes. \square

Remark 6.13. By a theorem of Cheeger and Gromov [CG86] the L^2 -Betti numbers of infinite amenable groups vanish. However, there are amenable groups Γ , e.g. the lamplighter group, such that $H_n(\Gamma; \mathcal{U}(\Gamma))$ does not vanish for some $n \geq 1$. See [LLS03]. In particular, the lamplighter group is *not* in \mathcal{CM} . We wonder whether the property of belonging to \mathcal{CM} is relevant to other spectral issues of amenable groups.

APPENDIX A. PRODUCT STRUCTURES ON TOR AND EXT

Let R be a ring, and be M, N R -modules. Then the Ext-groups $\text{Ext}_R^n(M, N)$ can be computed with a projective R -resolution P_* of M : $\text{Ext}_R^n(M, N) = H^n(\text{hom}_R(P_*, N))$. If M is a right R -module, we have $\text{Tor}_n^R(M, N) = H_n(P_* \otimes N)$. Notice that $H^n(\Gamma, M) = \text{Ext}_{R\Gamma}^n(R, M)$ and $H_n(\Gamma, M) = \text{Tor}_{R\Gamma}^n(R, M)$. We describe now a more *symmetric* way to define Ext- and Tor-groups which allows to introduce product structures.

We recollect some standard notations and facts about chain complexes (cf. [Bro94, chapter I, 0]). Let C_*, C'_* be non-negative chain complexes of left R -modules. Let $\text{hom}_R(C_*, C'_*)_n$, $n \geq 0$, be the abelian group of graded R -module homomorphisms of degree n from C_* to C'_* , i.e.

$$\text{hom}_R(C_*, C'_*)_n = \prod_{q \geq 0} \text{hom}_R(C_q, C'_{q+n}).$$

Then $\text{hom}_R(C_*, C'_*)_n$, $n \geq 0$, becomes a chain complex by the differential

$$\begin{aligned} \partial_{\text{hom}} : \text{hom}_R(C_*, C'_*)_n &\longrightarrow \text{hom}_R(C_*, C'_*)_{n-1} \\ \partial_{\text{hom}}(f) &= \partial f - (-1)^n f \partial. \end{aligned}$$

The 0-cycles of this complex are just the chain maps from C_* to C'_* , and the 0-boundaries are the nullhomotopic chain maps. Hence its zeroth homology is the abelian group of homotopy classes of chain maps from C_* to C'_* . If D_* is a complex of right R -modules, then $D_* \otimes_R C_*$ denotes the tensor product of complexes defined by

$$(D_* \otimes_R C_*)_n = \bigoplus_{i+j=n} D_i \otimes_R C_j$$

and equipped with the differential $\partial_{\otimes}(x \otimes y) = \partial x \otimes y + (-1)^{\deg x} x \otimes \partial y$.

Now let M and N be left R -modules, and be $M \leftarrow P_*$ and $N \leftarrow Q_*$ left projective R -resolutions. Consider N as a chain complex concentrated in degree zero. Due to [Bro94, theorem (8.5) on p. 29], the chain map $Q_* \rightarrow N$ induces a quasi-isomorphism $\text{hom}_R(P_*, Q_*)_* \rightarrow \text{hom}_R(P_*, N)_*$, i.e. the induced map in homology

$$H_{-n}(P_*, Q_*) \xrightarrow{\cong} H_{-n}(\text{hom}_R(P_*, N)) = H^n(\text{hom}_R(P_*, N)) = \text{Ext}_R^n(M, N)$$

is a (canonical) isomorphism. Similarly, if M is a right R -module and N is left R -module with right resp. left projective resolutions P_*, Q_* , then the chain map $Q_* \rightarrow N$ induces a quasi-isomorphism $P_* \otimes_R Q_* \rightarrow P_* \otimes N$, where the homology of the latter is $\text{Tor}_*^R(M, N)$. This is what we understand under the *symmetric way of defining Ext and Tor*.

Let us recall some **functoriality properties of the Ext and Tor**. Compare with [Gui80, p. 72-74]. Assume $R \subset S$ is a *flat ring extension*. Let M, N be R -modules (whether right

or left will be clear from the context) with projective R -resolutions P_* resp. Q_* . Then there are natural maps (*induction of scalars*)

$$(A.1) \quad \text{Ext}_R^n(N, M) \longrightarrow \text{Ext}_S^n(S \otimes_R N, S \otimes_R M),$$

$$(A.2) \quad \text{Tor}_n^R(N, M) \longrightarrow \text{Tor}_n^S(N \otimes_R S, S \otimes_R M).$$

In the symmetric picture above, (A.1) is induced by scalar induction

$$\text{hom}_R(Q_*, P_*)_n \longrightarrow \text{hom}_r(S \otimes_R Q_*, S \otimes_R P_*)_n, f \mapsto S \otimes_R f.$$

Similar for (A.2). If L is an S -module then we have – as a result of the adjunction isomorphism between induction and restriction – canonical isomorphisms

$$(A.3) \quad \text{Ext}_S^n(S \otimes_R M, L) \cong \text{Ext}_R^n(M, L)$$

$$(A.4) \quad \text{Tor}_n^S(M \otimes_R S, L) \cong \text{Tor}_n^R(M, L)$$

If p is a full idempotent in the ring R , implying that pRp and R are Morita equivalent (cf. remark 2.20), then we obtain natural isomorphisms

$$(A.5) \quad \text{Ext}_{pRp}^n(N, M) \xrightarrow{\cong} \text{Ext}_R^n(Rp \otimes_{pRp} N, Rp \otimes_{pRp} M)$$

$$(A.6) \quad \text{Tor}_n^{pRp}(N, M) \xrightarrow{\cong} \text{Tor}_n^R(N \otimes_{pRp} pR, Rp \otimes_{pRp} M)$$

In the symmetric picture, the map (A.5) is given by tensoring chain maps with Rp

$$\text{hom}_R(Q_*, P_*)_n \longrightarrow \text{hom}_R(Rp \otimes_{pRp} Q_*, Rp \otimes_{pRp} P_*)_n, f \mapsto Rp \otimes_{pRp} f.$$

Its inverse is given by tensoring with pR over R . Similar for Tor.

Next we introduce the multiplicative structures. Let M, N and L be R -modules with projective R -resolutions P_*, Q_* and W_* . The composition of chain maps in the symmetric picture induces homomorphisms

$$\text{Ext}_R^m(N, L) \otimes \text{Ext}_R^n(M, N) \longrightarrow \text{Ext}_R^{m+n}(M, L)$$

which turn $\text{Ext}_R^*(M, M)$ into a graded ring whose product is called **composition product**. Further, we have a product, called the **evaluation product**,

$$\text{Ext}_R^m(N, L) \otimes \text{Tor}_n^R(M, N) \longrightarrow \text{Tor}_{n-m}^R(M, L)$$

defined by

$$\text{hom}_R(Q_*, W_*)_m \otimes (P_* \otimes_R Q_*)_n \rightarrow (P_* \otimes Q_*)_{m-n}, f \otimes p \otimes q \rightarrow (-1)^{\deg f \deg p} p \otimes f(q).$$

Obviously, the homomorphisms in (A.1) and (A.5) are multiplicative.

Now assume R is *commutative*. Let M, N, L be $R\Gamma$ -modules, and let $c : M \otimes_R N \rightarrow L$ be a $R\Gamma$ -homomorphism, where $M \otimes_R N$ is equipped with the diagonal Γ -action. Then the **cup product** \cup and **cap product** \cap (see [Bro94, p. 109-117]) are maps

$$\cup : H^m(\Gamma, M) \otimes H^n(\Gamma, N) \longrightarrow H^{m+n}(\Gamma, M \otimes_R N) \xrightarrow{c} H^{m+n}(\Gamma, L)$$

$$\cap : H^m(\Gamma, M) \otimes H^n(\Gamma, N) \longrightarrow H_{m-n}(\Gamma, M \otimes_R N) \xrightarrow{c} H_{m-n}(\Gamma, L)$$

This yields ring and module structures in the cases $N = M = L$ and $N = L$. We record from [Bro94, theorem (4.6) on p. 115].

Lemma A.1. *The cup product and composition product resp. the cap product and evaluation product on $\text{Ext}_{R\Gamma}^*(R, R) = H^*(\Gamma, R)$, $\text{Tor}_*^{R\Gamma}(R, R) = H_*(\Gamma, R)$ coincide.*

Let Y be a compact topological space on which a group Γ acts. Write $\mathcal{F}(Y) = \mathcal{F}(Y; R)$. By lemma 2.18 $R\Gamma \subset \mathcal{F}(Y) \rtimes \Gamma$ is a flat ring extension. On $H^*(\Gamma, \mathcal{F}(Y))$ we can exhibit two product structures. The first one is the cup product coming from $\mathcal{F}(Y) \otimes_R \mathcal{F}(Y) \rightarrow \mathcal{F}(Y)$, $f \otimes g \mapsto fg$, and the second one is the composition product coming from the isomorphism

$$H^n(\Gamma, \mathcal{F}(Y)) = \text{Ext}_{R\Gamma}^n(R, \mathcal{F}(Y)) \xrightarrow[\text{(A.3)}]{\cong} \text{Ext}_{\mathcal{F}(Y) \rtimes \Gamma}^n(\mathcal{F}(Y), \mathcal{F}(Y)).$$

Similarly, using (A.4) we obtain a cap and an evaluation product on $H^*(\Gamma, \mathcal{F}(Y))$ and $H_*(\Gamma, \mathcal{F}(Y))$. Then the following fact follows again from [Bro94, theorem (4.6) on p. 115].

Lemma A.2. *The cup and composition resp. the cap and evaluation product on $H^*(\Gamma, \mathcal{F}(Y))$, $H_*(\Gamma, \mathcal{F}(Y))$ coincide.*

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